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# Notes on Mathematical Topics in Fluid Mechanics and Applications 

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## Preface

These notes have been mostly written by Emmanuele DiBenedetto. Indeed, Emmanuele authored the first chapter (which is basically an excerpt of his wider monograph Classical Mechanics, published by Birkhäuser) and most of the second one; later, I revised and finished the work, after Emmanuele passed away in May 2021.

Our common aim was to concentrate on the main results, at the same time giving a brief introduction to the most important and interesting open problems. The list of references is meant to help interested students, so that they can access the original works, and also get to know problems that here are only very briefly sketched.

I am grateful to Giulia Cavalleri, Matteo Ferrari, Bruno Minniti, Edoardo Tolotti who followed this course in Spring 2023 for their valuable comments, and for the patience they had with me. I am indebted to prof. Ermanno Gherardi, who insisted that I should teach a Math course at Collegio Volta, where he is presently Master: without Ermanno, there would have been no course. Finally, the kind support by Collegio Volta staff has helped a lot.
University of Pavia, Ugo Gianazza,
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## PHYSICS OF THE NAVIER-STOKES EQUATIONS

## 1 Geometry of Deformations

A bounded open, connected set in $E_{o} \subset \mathbb{R}^{N}$ deforms in time to $E$ in the sense that points $y \in E_{o}$ are in one-to-one correspondence with points $x \in E$ through smooth, non-intersecting trajectories $t \rightarrow x(t)$ such that $x(0)=y$ and $x(t)=x$. This defines a flow map and a velocity field

$$
\begin{equation*}
x=\Phi(y, t), \quad \mathbf{v}(x, t)=\Phi_{t}(y, t) . \tag{1.1}
\end{equation*}
$$

The functions $\Phi(\cdot, t)$ may be regarded as a family of transformations defined in $E_{o}$ and parametrized with $t$. These transformations will be assumed to be smooth and invertible independent of $t$. In the Lagrangian formalism, kinematic informations on $x(t) \in E$ are provided by the trajectories $t \rightarrow x(t)$ independently of their membership to $E$, as an open connected subset of $\mathbb{R}^{N}$, this bearing a role only in the determination of such paths ([17]). In the Eulerian formalism, kinematic informations on points $x \in E$ are provided by the flow map $\Phi(\cdot, t)$, which bears the "globality" of $E_{o}$ and $E([9,10])$. In both formalisms these quantities must coincide. Therefore $\dot{x}=\mathrm{v}(x, t)$ and

$$
\begin{equation*}
\ddot{x}=\frac{d}{d t} \dot{x}=\frac{\partial}{\partial t} \mathbf{v}(x, t)+\dot{x} \cdot \nabla_{x} \mathbf{v}=D_{t} \mathbf{v} \tag{1.2}
\end{equation*}
$$

where the operator $D_{t}$ formally defined by

$$
\begin{equation*}
D_{t}=\frac{\partial}{\partial t}+\mathbf{v} \cdot \nabla_{x} \tag{1.3}
\end{equation*}
$$

is the total or material derivative along Lagrangian paths. For $t$ fixed, the Jacobian of the transformation $\Phi(\cdot, t)$ is

$$
J(x, t)=J[\Phi(y, t)]=\operatorname{det}\left(\frac{\partial \Phi_{i}(y, t)}{\partial y_{j}}\right)=A_{i j} \frac{\partial \Phi_{k}(y, t)}{\partial y_{j}} \delta_{i k}
$$

where $A_{i j}$ is the determinant of the algebraic complement of the $(i j)$ th entry of the Jacobian matrix $\nabla \Phi$.

Here, and in the sequel of this first chapter, summation over repeated indices is assumed.

Proposition 1.1 (Euler [9]) $J_{t}=J \operatorname{div} v$.
Proof. From the previous expression of $J^{1}$

$$
\begin{aligned}
J_{t}(x, t) & =\frac{\partial}{\partial t} \operatorname{det}\left(\frac{\partial \Phi_{i}(y, t)}{\partial y_{j}}\right)=A_{i j} \frac{\partial}{\partial y_{j}} \frac{\partial \Phi_{i}(y, t)}{\partial t} \\
& =A_{i j} \frac{\partial v_{i}}{\partial y_{j}}=A_{i j} \frac{\partial v_{i}}{\partial x_{k}} \frac{\partial \Phi_{k}(y, t)}{\partial y_{j}}=\frac{\partial v_{i}}{\partial x_{i}} J .
\end{aligned}
$$

### 1.1 Incompressible Deformations

If an infinitesimal portion about any $y \in E_{o}$ moves by possibly changing its shape and/or configuration, but keeping fixed its infinitesimal volume, then $J_{t}=0$ and consequently $\operatorname{div} \mathbf{v}=0$ and the deformation is called incompressible. Vice versa, a deformation is incompressible if and only if $\operatorname{div} \mathbf{v}=0$.

### 1.2 The Equation of Continuity

Let $G \subset \mathbb{R}^{N}$ be open and let $E(t) \subset G$ be a deforming sub-domain of $G$ with smooth boundary $\partial E(t)$. For a smooth function $(x, t) \rightarrow \rho(x, t)$ defined in a neighborhood of $E(t)$, by the previous proposition

$$
\begin{aligned}
\frac{d}{d t} \int_{E(t)} \rho(x, t) d x & =\frac{d}{d t} \int_{E_{o}} \rho(\Phi(y, t), t) J d y \\
& =\int_{E_{o}} \frac{d}{d t}[\rho(\Phi(y, t), t) J] d y \\
& =\int_{E_{o}}\left[\left(\Phi_{t} \cdot \nabla_{x} \rho+\rho_{t}\right) J+\rho J_{t}\right] d y \\
& =\int_{E_{o}}\left[\rho_{t}+\operatorname{div}(\rho \mathbf{v})\right] J d y \\
& =\int_{E(t)}\left[\rho_{t}+\operatorname{div}(\rho \mathbf{v})\right] d x
\end{aligned}
$$

If $\rho(x, t)$ is the material density of a body occupying the domain $G$, then for every deforming subset $E(t) \subset G$

[^0]$$
\int_{E(t)} \rho(x, t) d x=\text { mass of the body in } E(t)
$$

If elements of $G$ evolve conserving their mass, then

$$
\frac{d}{d t} \int_{E(t)} \rho(x, t) d x=0
$$

for all deforming sub-domains $E(t) \subset G$. Since $E(t) \subset G$ is arbitrary, local deformations of $G$ preserve the mass if and only if

$$
\rho_{t}+\operatorname{div}(\rho \mathbf{v})=0 \quad \text { pointwise in } G .
$$

This is the continuity equation and expresses conservation of mass.

## 2 Cardinal Equations

Whereas in the previous section we were working in $\mathbb{R}^{N}$ with $N \geq 2$, now we choose $N=3$. Along the motion, points $x \in E \subset G$ are acted upon by a material distributions of forces $\mathbf{f}(x, \dot{x}, t) \rho(x, t) d x$ (f is a specific force, that is, force per unit mass), and by reactions acting on $\partial E$ due to the remaining portion $G-E$ which opposes the possible deformation of $E$. These are apriori unknown, depend on the material structure of $G$, and should not depend on the particular sub-domain $E \subset G$. In the Cauchy formalism they are represented by a smooth vector-valued function

$$
G \times S_{1} \times \mathbb{R} \ni(x, \mathbf{n}, t) \rightarrow \mathbf{T}(x, \mathbf{n}, t) \in \mathbb{R}^{3}
$$

where $S_{1}$ is the unit sphere in $\mathbb{R}^{3}$. Then, assuming that $\partial E$ is smooth, reaction forces of $G-E$, acting on $\partial E$ are described by

$$
\{\text { reactions opposing deformations of } E\}=\int_{\partial E} \mathbf{T}(x, \mathbf{n}, t) d \sigma
$$

where $d \sigma$ is the surface measure on $\partial E$ and $\mathbf{n}$ is the outward unit normal to $\partial E$ at $x \in \partial E$. The component $(\mathbf{T} \cdot \mathbf{n}) \mathbf{n}$ of $\mathbf{T}$ along $\mathbf{n}$ is the traction or compression force, whereas the component $\mathbf{T}-(\mathbf{T} \cdot \mathbf{n}) \mathbf{n}$ tangent to $\partial E$ at $x$ is the shear force. By d'Alembert principle the motion of any sub-domain $E \subset G$ is a sequence of instantaneous equilibrium states, parameterized with time, of all forces acting on that portion, including the reactions to deformation. Thus,

$$
\begin{align*}
\int_{E}[\ddot{x}-\mathbf{f}(x, \dot{x}, t)] \rho d x & =\int_{\partial E} \mathbf{T}(x, \mathbf{n}, t) d \sigma  \tag{2.1}\\
\int_{E} x \wedge[\ddot{x}-\mathbf{f}(x, \dot{x}, t)] \rho d x & =\int_{\partial E} x \wedge \mathbf{T}(x, \mathbf{n}, t) d \sigma \tag{2.2}
\end{align*}
$$

for all sub-domains $E \subset G$.

Lemma 2.1 $\mathbf{T}(\cdot, \mathbf{n}, t)=-\mathbf{T}(\cdot,-\mathbf{n}, t)$.
Proof. Fix $P \in G$ and $\mathbf{n} \in S_{1}$. For $0<\varepsilon, \delta \ll 1$ consider the disc $D_{\varepsilon}(P)$ centered at $P$ and radius $\varepsilon$, normal to $\mathbf{n}$, and the right cylinder $C_{\delta}(P)$ of base $D_{\varepsilon}(P)$ and height $\delta$. Write (2.1) over $C_{\delta}(P)$ and let $\delta \rightarrow 0$ by keeping $\varepsilon>0$ fixed, to obtain

$$
\int_{D_{\varepsilon}(P)} \mathbf{T}(x, \mathbf{n}, t) d \sigma=-\int_{D_{\varepsilon}(P)} \mathbf{T}(x,-\mathbf{n}, t) d \sigma
$$

Divide both sides by $\left|D_{\varepsilon}(P)\right|$ and let $\varepsilon \rightarrow 0$.

## 3 The Stress Tensor and Cauchy's Theorem

Having fixed a triad $\Sigma=\left\{O ; \mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}\right\}$, represent $\mathbf{n} \in S_{1}$ by its director cosines $\mathbf{n}=\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right)$ with respect to the coordinate axes of $\Sigma$.
Theorem 3.1 (Cauchy). For all $\mathbf{n}=\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right) \in S_{1}$

$$
\mathbf{T}(\cdot, \mathbf{n}, t)=\alpha_{i} \mathbf{T}\left(\cdot, \mathbf{e}_{i}, t\right)
$$

Proof. Fix $P \in G$ and $\mathbf{n} \in S_{1}$ and write down (2.1) where $E$ is the tetrahedron with vertex in $P$, height $0<\varepsilon \ll 1$, base $\triangle A B C$ normal to $\mathbf{n}$, and faces $\triangle A P B, \triangle B P C, \triangle C P A$, parallel to the coordinate planes. By setting $\Delta \sigma=$ $|\triangle A B C|$, one has $|E|=\frac{1}{3} \varepsilon \Delta \sigma$, and

$$
|\Delta A P B|=\alpha_{3} \Delta \sigma, \quad|\Delta B P C|=\alpha_{1} \Delta \sigma, \quad|\Delta A P C|=\alpha_{2} \Delta \sigma
$$

For these choices, (2.1) takes the form

$$
\begin{aligned}
\int_{E}[\ddot{x}-\mathbf{f}(x, \dot{x}, t)] \rho d x= & \int_{\triangle A B C} \mathbf{T}(x, \mathbf{n}, t) d \sigma+\int_{\triangle B P C} \mathbf{T}\left(x,-\mathbf{e}_{1}, t\right) d \sigma \\
& +\int_{\triangle A P C} \mathbf{T}\left(x,-\mathbf{e}_{2}, t\right) d \sigma+\int_{\triangle A P B} \mathbf{T}\left(x,-\mathbf{e}_{3}, t\right) d \sigma
\end{aligned}
$$

Dividing both sides by $\Delta \sigma$, gives

$$
\begin{aligned}
\frac{\varepsilon}{3|E(t)|} \int_{E}[\ddot{x}-\mathbf{f}(x, \dot{x}, t)] \rho d x= & \frac{1}{|\Delta A B C|} \int_{\Delta A B C} \mathbf{T}(x, \mathbf{n}, t) d \sigma \\
& +\frac{\alpha_{1}}{|\Delta B P C|} \int_{\Delta B P C} \mathbf{T}\left(x,-\mathbf{e}_{1}, t\right) d \sigma \\
& +\frac{\alpha_{2}}{|\Delta A P C|} \int_{\Delta A P C} \mathbf{T}\left(x,-\mathbf{e}_{2}, t\right) d \sigma \\
& +\frac{\alpha_{3}}{|\triangle A P B|} \int_{\triangle A P B} \mathbf{T}\left(x,-\mathbf{e}_{3}, t\right) d \sigma
\end{aligned}
$$

Let $\varepsilon \rightarrow 0$ by keeping the vertex $P$ of the tetrahedron fixed and the base $\triangle A B C$ normal to $\mathbf{n}$.

While computed at $\mathbf{e}_{i}$, the vectors $\mathbf{T}\left(\cdot, \mathbf{e}_{i}, t\right)$, need not be directed along the homologous coordinate axes. The components $\tau_{i j}(\cdot, t)=\mathbf{T}\left(\cdot, \mathbf{e}_{j}, t\right) \cdot \mathbf{e}_{i}$ of $\mathbf{T}\left(\cdot, \mathbf{e}_{j}, t\right)$ along $\mathbf{e}_{i}$, define a matrix

$$
\mathbb{T}=\left(\tau_{i j}\right)=\left(\begin{array}{lll}
\tau_{11} & \tau_{12} & \tau_{13} \\
\tau_{21} & \tau_{22} & \tau_{23} \\
\tau_{31} & \tau_{32} & \tau_{33}
\end{array}\right)
$$

called stress tensor. The entries $\tau_{i i}$ are traction or compression stresses and $\tau_{i j}$, for $i \neq j$ are shear stresses. The shear force acting on an infinitesimal plane surface normal to $\mathbf{e}_{1}$ is $\tau_{21} \mathbf{e}_{2}+\tau_{31} \mathbf{e}_{3}$. In general

$$
\left\{\text { shear force relative to } \mathbf{e}_{i}\right\}=\sum_{j \neq i} \tau_{j i} \mathbf{e}_{j}
$$

Corollary 3.1 $\mathbf{T}(\cdot, \mathbf{n}, t)=\mathbb{T} \cdot \mathbf{n}=\left(\tau_{i j}\right) \mathbf{n}$.
Proof. From the definitions and Theorem 3.1

$$
\begin{aligned}
& \mathbf{T}(\cdot, \mathbf{n}, t)=\alpha_{j} \mathbf{T}\left(\cdot, \mathbf{e}_{j}, t\right)=\alpha_{j}\left[\mathbf{T}\left(\cdot, \mathbf{e}_{j} ; t\right) \cdot \mathbf{e}_{i}\right] \mathbf{e}_{i} \\
& =\alpha_{1}\left(\begin{array}{l}
\tau_{11} \\
\tau_{21} \\
\tau_{31}
\end{array}\right)+\alpha_{2}\left(\begin{array}{c}
\tau_{12} \\
\tau_{22} \\
\tau_{32}
\end{array}\right)+\alpha_{3}\left(\begin{array}{c}
\tau_{13} \\
\tau_{23} \\
\tau_{33}
\end{array}\right)=\left(\begin{array}{lll}
\tau_{11} & \tau_{12} & \tau_{13} \\
\tau_{21} & \tau_{22} & \tau_{23} \\
\tau_{31} & \tau_{32} & \tau_{33}
\end{array}\right)\left(\begin{array}{l}
\alpha_{1} \\
\alpha_{2} \\
\alpha_{3}
\end{array}\right)=\mathbb{T} \cdot \mathbf{n} .
\end{aligned}
$$

Corollary 3.2 Let $G \subset \mathbb{R}^{3}$ be an open set identified with a material system of density $\rho(\cdot, t)$, whose points $x \in G$ are in motion under the external force density $\mathbf{f}(x, \dot{x}, t)$ and the internal stress tensor $\mathbb{T}$. Then

$$
\begin{equation*}
[\ddot{x}-\mathbf{f}(x, \dot{x}, t)] \rho=\operatorname{div} \mathbb{T} \quad \text { in } \quad G . \tag{3.1}
\end{equation*}
$$

Proof. Let $E$ be any portion of $G$ with smooth boundary $\partial E$. By the GaussGreen theorem and Corollary 3.1

$$
\int_{\partial E} \mathbf{T}(x, \mathbf{n}, t) d \sigma=\int_{\partial E} \mathbb{T} \cdot \mathbf{n} d \sigma=\int_{E} \operatorname{div} \mathbb{T} d x
$$

Therefore, (2.1) takes the form

$$
\int_{E}[\ddot{x}-\mathbf{f}(x, \dot{x}, t)] \rho d x=\int_{E} \operatorname{div} \mathbb{T} d x
$$

for all sub-domains $E \subset G$.

### 3.1 Symmetry of the Stress Tensor

Proposition $3.1\left(\tau_{i j}\right)=\left(\tau_{j i}\right)$.
Proof. By the Gauss-Green theorem

$$
\begin{aligned}
\int_{\partial E} x \wedge \mathbf{T}(x, \mathbf{n}, t) d \sigma & =\int_{\partial E} x_{h} \tau_{i j} \mathbf{e}_{h} \wedge \mathbf{e}_{i} \alpha_{j} d \sigma \\
& =\int_{E} \frac{\partial}{\partial x_{j}}\left(\tau_{i j} x_{h}\right) \mathbf{e}_{h} \wedge \mathbf{e}_{i} d x \\
& =\int_{E} \frac{\partial \tau_{i j}}{\partial x_{j}} x_{h} \mathbf{e}_{h} \wedge \mathbf{e}_{i} d x+\int_{E} \tau_{i j} \delta_{h j} \mathbf{e}_{h} \wedge \mathbf{e}_{i} d x \\
& =\int_{E} x \wedge \operatorname{div} \mathbb{T} d x-\int_{E} \tau_{i j} \mathbf{e}_{i} \wedge \mathbf{e}_{j} d x
\end{aligned}
$$

Put this in the second cardinal equation (2.2) and take into account (3.1) to obtain

$$
\int_{E} \tau_{i j} \mathbf{e}_{i} \wedge \mathbf{e}_{j} d x=\int_{E}\left[\left(\tau_{23}-\tau_{32}\right) \mathbf{e}_{1}+\left(\tau_{31}-\tau_{13}\right) \mathbf{e}_{2}+\left(\tau_{12}-\tau_{21}\right) \mathbf{e}_{3}\right] d x=0
$$

for all sub-domains $E \subset G$.

### 3.2 Miscellaneous Remarks

The matrix $\mathbb{T}$ is intrinsic to the system and independent of its representations in the following sense. Let $\Sigma^{\prime}=\left\{O ; \mathbf{e}_{1}^{\prime}, \mathbf{e}_{2}^{\prime}, \mathbf{e}_{3}^{\prime}\right\}$ be a new triad obtained by $\Sigma$ by a rotation of the coordinate axes realized by a unitary matrix $U$, so that in particular $\mathbf{e}_{j}^{\prime}=U \mathbf{e}_{j}$. By Corollary 3.1

$$
\tau_{i j}^{\prime}=\mathbf{T}\left(\cdot, \mathbf{e}_{j}^{\prime}, t\right) \cdot \mathbf{e}_{i}^{\prime}=\left(\tau_{h k}\right) U \mathbf{e}_{j} \cdot U \mathbf{e}_{i}=\mathbf{e}_{i}^{t}\left[U^{t}\left(\tau_{h k}\right) U\right] \mathbf{e}_{j}=\left[U^{t}\left(\tau_{h k}\right) U\right]_{i j}
$$

The tensor $\mathbb{T}$ is a linear map in $\mathbb{R}^{3}$ whose matrix $\left(\tau_{i j}\right)$ is a representative. We will call stress tensor both $\mathbb{T}$ and its matrix representations.

The unknowns of the motion are the trajectories $t \rightarrow x(t)$ of the points of $G$, the density function $\rho(\cdot, t)$ and the 9 components $\tau_{i j}$ of $\mathbb{T}$. The second cardinal equation (2.2), which amounts to 3 scalar equations, has been used to establish the symmetry of $\mathbb{T}$ and thus reduce by 3 the unknowns of the motions. The remaining first cardinal equation, in the pointwise form (3.1), amounts to 3 scalar equations, which alone are insufficient to resolve the motion. One needs to provide additional information on the material structure and on the tensorial state of the system both in the interior of $G$ and on its boundary $\partial G$. For example for rigid systems $\rho=$ const, and the $\mathbf{T}$ is the rigidity constraint.

## 4 Perfect Fluids and Cardinal Equations

A fluid is a continuum material system, whose equilibrium configurations are possible if and only if the stress tensor $\mathbb{T}$ is proportional to the identity $\mathbb{I}$, that is if

$$
\mathbf{T}(\cdot, \mathbf{n})=\mathbb{T} \cdot \mathbf{n}=-p(\cdot) \mathbf{n} \quad \text { in steady state }
$$

where $p(\cdot)$ is a smooth function defined in $G$, called pressure. This formula is a mathematical rendering of the Pascal principle, by which the pressure in any point of the fluid, exerts equal force by unit surface, in all directions ([34]). This mathematical definition of fluid reflects the intuitive idea of a material continuum system that does not oppose the mutual sliding of its ideal internal layers. If in the fluid at rest the shear components of its stress tensor were not zero, these would generate an incipient shearing of internal layers, since the system does not have a mechanism to oppose it. Likewise, an ideal material surface traced in the fluid at rest, remains in equilibrium only if acted upon by forces normal to it. In a real fluid in motion, the kinematic viscosity generates shear stresses that oppose layer sliding. Then real fluids are classified in more viscous (oil, paraffin, etc.) and less viscous (alcohol, ether, gas, etc.) according to the size of these shear stresses. A real fluid is ideal or perfect if the shear stresses are negligible even in dynamic regime, that is if

$$
\begin{equation*}
\mathbf{T}(\cdot, \mathbf{n}, t)=\mathbb{T} \cdot \mathbf{n}=-p(\cdot, t) \mathbf{n} \quad \text { in } G \text { and for all times. } \tag{4.1}
\end{equation*}
$$

In such a case $\operatorname{div} \mathbb{T}=-\nabla p(\cdot, t)$ and (3.1) takes the form

$$
\begin{equation*}
\rho[\ddot{x}-\mathbf{f}(x, \dot{x}, t)]+\nabla p=0 \quad \text { in } G \text { for all } t \tag{4.2}
\end{equation*}
$$

Equation (4.1) is the constitutive law of ideal fluids and (4.2) are the cardinal or the momentum equations of an ideal fluid.

## 5 Rotations and Deformations

Let $\mathbf{v}(\cdot, t)$ be the velocity field generated by the flow map in (1.1) and assume that the fluid at time $t$ undergoes an elemental rigid motion of characteristics $\mathbf{v}\left(x_{o}, t\right)$ and $\boldsymbol{\omega}$, where $x_{o}$ is an arbitrary, but fixed point in the instantaneously rigid fluid. By the Poisson formula

$$
\mathbf{v}(x, t)=\mathbf{v}\left(x_{o}, t\right)+\boldsymbol{\omega} \wedge\left(x-x_{o}\right) .
$$

Since the motion is instantaneously rigid, $\boldsymbol{\omega}$ does not depend on the variables $x$ of the generic point in the fluid. Taking the curl of both sides gives

$$
\begin{equation*}
\operatorname{curl} \mathbf{v}=\left(v_{3, x_{2}}-v_{2, x_{3}}\right) \mathbf{e}_{1}+\left(v_{1, x_{3}}-v_{3, x_{1}}\right) \mathbf{e}_{2}+\left(v_{1, x_{2}}-v_{1, x_{2}}\right) \mathbf{e}_{3}=2 \boldsymbol{\omega} \tag{5.1}
\end{equation*}
$$

Therefore, curl $\mathbf{v}(x, t)$ gives, apart from the factor 2 , the angular velocity of the infinitesimal element of fluid about $x$, regarded as instantaneously rigid.

For this reason curl $\mathbf{v}(\cdot, t)$ is called vorticity field. If $\operatorname{curl} \mathbf{v}(\cdot, t)=0$ the field is irrotational. If $G$ is simply connected, an irrotational field is also potential, that is there exists a function $\varphi(\cdot, t) \in C^{1}(G)$, called kinetic potential, such that $\mathbf{v}(\cdot, t)=\nabla \varphi(\cdot, t)$. The flow is called potential, and the velocity field $\mathbf{v}(\cdot, t)$ is normal to the instantaneous equi-potential $\operatorname{surfaces}[\varphi(\cdot, t)=\operatorname{const}(t)]$. If the velocity field is stationary, the kinetic potential is independent of $t$ and the trajectories of the fluid particles are normal to the equi-potential surfaces.

Next expand $\mathbf{v}(\cdot, t)$ in Taylor series about a point $x_{o}$ in the fluid, to obtain

$$
\mathbf{v}(x, t)=\mathbf{v}\left(x_{o}, t\right)+\left[\nabla \mathbf{v}\left(x_{o}, t\right)\right] \cdot\left(x-x_{o}\right)+\mathbf{o}\left(\left|x-x_{o}\right|^{2}\right)
$$

Therefore, up to terms of higher order

$$
v_{i}(x, t)=v_{i}\left(x_{o}, t\right)+v_{i, x_{j}}\left(x_{j}-x_{o, j}\right), \quad i=1,2,3 .
$$

For fixed indices $i, j$

$$
\begin{equation*}
v_{i, x_{j}}=\frac{1}{2}\left(\frac{\partial v_{i}}{\partial x_{j}}+\frac{\partial v_{j}}{\partial x_{i}}\right)+\frac{1}{2}\left(\frac{\partial v_{i}}{\partial x_{j}}-\frac{\partial v_{j}}{\partial x_{i}}\right)=\mathcal{D}_{i j}+\mathcal{R}_{i j} \tag{5.2}
\end{equation*}
$$

The entries $\mathcal{D}_{i j}$ e $\mathcal{R}_{i j}$ define two tensors $\mathcal{D}$ and $\mathcal{R}$. The first is symmetric and is called deformation tensor. The second is skew-symmetric and is called rotation tensor. With this notation, the previous Taylor expansion takes the approximate form

$$
\begin{equation*}
\mathbf{v}(x, t)=\mathbf{v}\left(x_{o}, t\right)+\mathcal{D} \cdot\left(x-x_{o}\right)+\mathcal{R} \cdot\left(x-x_{o}\right) \tag{5.3}
\end{equation*}
$$

Consider an infinitesimal arc $d\left(x-x_{o}\right)$ within the fluid and along its motion, of length $d \ell=\sqrt{d\left(x-x_{o}\right)^{2}}$. Then

$$
\begin{aligned}
\frac{d}{d t} d \ell^{2} & =2 d\left(x-x_{o}\right) \cdot d\left(\dot{x}-\dot{x}_{o}\right) \\
& =d\left(x_{i}-x_{o, i}\right) v_{i, x_{j}} d\left(x_{j}-x_{o, j}\right) \\
& =2 d\left(x-x_{o}\right)^{t} \cdot \mathcal{D} \cdot d\left(x-x_{o}\right)
\end{aligned}
$$

Therefore, $\mathcal{D}$ tracks the deformations of infinitesimal lengths along the motion. In a rigid motion lengths are preserved and $\mathcal{D}=0$. If $\mathcal{D}=\lambda \mathbb{I}$ then the deformation occurs uniformly along the coordinate axes and the fluid expands if $\lambda>0$ and contracts if $\lambda<0$. From the definition of $\mathcal{R}$

$$
\mathcal{R} \cdot\left(x-x_{o}\right)=\frac{1}{2} \operatorname{curl} \mathbf{v} \wedge\left(x-x_{o}\right)=\boldsymbol{\omega} \wedge\left(x-x_{o}\right)
$$

Hence, $\mathcal{R}$ gives the angular velocity of the system as if it were in instantaneous rigid motion. These remarks and (5.3) suggest we regard the infinitesimal motion of a fluid as the sum of

1. an infinitesimal translation along $\mathbf{v}\left(x_{o}, t\right)$;
2. an infinitesimal deformation along the coordinate axes;
3. an infinitesimal rigid rotation about the axis through $x_{o}$ and directed as $\operatorname{curl} \mathbf{v}\left(x_{o}, t\right)$.

This is known as the Cauchy theorem.

## 6 Friction Tensor for Newtonian Viscous Fluids

In real fluids the friction generated by the mutual sliding of infinitesimal layers generates shear forces that oppose the motion. The stress tensor $\mathbb{T}$ takes the more general form

$$
\begin{equation*}
\tau_{i j}=-p \delta_{i j}+\sigma_{i j} \tag{6.1}
\end{equation*}
$$

where $\sigma_{i j}$ are due to friction. Two infinitesimal layers slide over one another if their velocity is different. Therefore $\sigma_{i j}=\sigma_{i j}(\nabla \mathbf{v})$ depend on the gradient of the velocity. Moreover, $\sigma_{i j}=0$ if $\nabla \mathbf{v}=0$. Assuming that $\sigma_{i j}(\cdot)$ are smooth functions of their arguments, they can be expanded in Taylor's series about the origin of their arguments to give

$$
\sigma_{i j}(\nabla \mathbf{v})=\gamma_{i j h k} v_{h, x_{k}}+o_{i j}\left(\|\nabla \mathbf{v}\|^{2}\right), \quad \text { where } \quad \gamma_{i j h k}=\left.\frac{\partial \sigma_{i j}}{\partial v_{h, x_{k}}}\right|_{\nabla \mathbf{v}=0}
$$

for $i, j=1,2,3$, where $o_{i j}(\cdot)$ are infinitesimal of higher order in $|\nabla \mathbf{v}|$. A fluid is Newtonian if $\left(\sigma_{i j}\right)$ depends linearly on $\nabla \mathbf{v}$ so that the higher order terms in the previous Taylor expansions are negligible. Water and alcohol are Newtonian, whereas paints and gels are not.

The numbers $\gamma_{i j h k}$ as the indices $i, j, h, k$ run over $1,2,3$, represent a 4 thorder tensor which quantifies the stresses due to the presence of internal friction in a fluid. By its physical nature such a tensor must be isotropic, that is must be independent of rotations of the Cartesian system of its representation.
Lemma 6.1 Let ( $\gamma_{i j h k}$ ) for $i, j, h, k=1,2,3$ be a representation of an isotropic tensor $\sigma$. Then there exists numbers $\lambda$ and $\mu_{1}, \mu_{2}$, such that

$$
\gamma_{i j h k}=\lambda \delta_{i j} \delta_{h k}+\mu_{1} \delta_{i h} \delta_{j k}+\mu_{2} \delta_{i k} \delta_{j h}
$$

The lemma is established in $\S 8 .{ }^{2}$ Assuming it for the moment, it implies that $\sigma_{i j}$ must be of the form

$$
\sigma_{i j}=\lambda \delta_{i j} v_{h, x_{h}}+\mu_{1} v_{i, x_{j}}+\mu_{2} v_{j, x_{i}}
$$

Since $\left(\sigma_{i j}\right)$ must also be symmetric (Proposition 3.1)

$$
\sigma_{i j}=\lambda \delta_{i j} v_{h, x_{h}}+\mu_{1} v_{j, x_{i}}+\mu_{2} v_{i, x_{j}}
$$

[^1]Adding these two expressions of $\sigma_{i j}$ gives

$$
\sigma_{i j}=\frac{1}{2} \lambda \operatorname{div} \mathbf{v} \delta_{i j}+\frac{1}{2} \bar{\mu}\left(v_{i, x_{j}}+v_{j, x_{i}}\right)=\frac{1}{2} \lambda \operatorname{div} \mathbf{v} \delta_{i j}+\frac{1}{2} \bar{\mu} \mathcal{D}_{i j},
$$

where $\bar{\mu}=\mu_{1}+\mu_{2}$ and $\left(\mathcal{D}_{i j}\right)$ is the deformation tensor introduced in (5.2). This representation of the friction stress tensor in Newtonian fluids is due to Stokes ([48]). If the fluid is also incompressible

$$
\begin{equation*}
\sigma_{i j}=\frac{1}{2} \bar{\mu} \mathcal{D}_{i j} . \tag{6.2}
\end{equation*}
$$

The constant $\frac{1}{2} \bar{\mu}$ is called the kinematic viscosity, and is determined experimentally. By thermodynamics considerations $\bar{\mu}>0$ ([25] page 213, and [42], Chapter V).

## 7 The Navier-Stokes Equations

A Newtonian, viscous, incompressible fluid moves in a domain $G \subset \mathbb{R}^{3}$. The momentum equations for such a fluid are those in (3.1). Taking into account the form of the acceleration $\ddot{x}$, and the form (6.1)-(6.2) of the stress tensor $\mathbb{T}$, these equations take the form

$$
\left[\mathbf{v}_{t}+(\mathbf{v} \cdot \nabla) \mathbf{v}-f\right] \rho+\nabla p=\frac{1}{2} \bar{\mu} \operatorname{div}\left(\mathcal{D}_{i j}\right)=\bar{\mu} \Delta \mathbf{v}+\nabla \operatorname{div} \mathbf{v}
$$

Therefore, since the fluid is incompressible

$$
\begin{array}{ll}
\mathbf{v}_{t}-\mu \Delta \mathbf{v}+(\mathbf{v} \cdot \nabla) \mathbf{v}+\frac{1}{\rho} \nabla p=f &  \tag{7.1}\\
\operatorname{div} \mathbf{v}=0 & \text { in } G \times \mathbb{R}^{+}
\end{array}
$$

where $\mu=\bar{\mu} / \rho$. We rename the constant $\mu$ as the kinematic viscosity and its physical dimensions are length-squared over time.

### 7.1 Conservation and Dissipation of Energy

Assume that there are no external forces, so that $\mathbf{f}=0$. Multiply (7.1) by $\mathbf{v}$ and perform standard vector calculus operations, to get

$$
\begin{equation*}
D_{t} \mathcal{B}=\frac{p_{t}}{\rho}+\mu \Delta \frac{1}{2}|\mathbf{v}|^{2}-\mu \sum_{i=1}^{3}\left|\nabla v_{i}\right|^{2} \quad \text { where } \mathcal{B}=\frac{1}{2}|\mathbf{v}|^{2}+\frac{p}{\rho} \tag{7.2}
\end{equation*}
$$

The term $\mathcal{B}$ is the specific energy of a material particle about $x$. Therefore, the left-hand side of (7.2) is the material derivative of such a specific energy.

The first term on the right-hand side is the time-variation of the internal energy about $x$. The second term can be regarded as a dissipation of kinetic energy due to viscosity. The last term is the energy dissipation due to the rough mutual sliding of infinitesimal layers one over one another. Thus, the variation of energy along Lagrangian paths is balanced by the time-variation of the internal energy and the dissipation of energy due to viscosity.

### 7.2 Dimensionless Formulation, Reynolds Number and Similarities

The Navier-Stokes equations (7.1) are written in their physical dimensions. To render them dimensionless select length and time units $\ell$ and $\tau$ and introduce dimensionless variables and quantities ${ }^{3}$

$$
x^{\prime}=\frac{x}{\ell}, \quad t^{\prime}=\frac{t}{\tau}, \quad \mathbf{v}^{\prime}=\frac{\tau}{\ell} \mathbf{v}, \quad p^{\prime}=\frac{\tau^{2}}{\ell^{2}} \frac{p}{\rho}, \quad \mathbf{f}^{\prime}=\frac{\tau^{2}}{\ell} \mathbf{f} .
$$

Then (7.1) become

$$
\begin{align*}
\mathbf{v}_{t^{\prime}}^{\prime}-\frac{1}{R} \Delta^{\prime} \mathbf{v}^{\prime}+\left(\mathbf{v}^{\prime} \cdot \nabla^{\prime}\right) \mathbf{v}^{\prime}+\nabla^{\prime} p^{\prime} & =\mathbf{f}^{\prime} \quad \text { in } G^{\prime} \times \mathbb{R}^{\prime+}  \tag{7.3}\\
\operatorname{div}^{\prime} \mathbf{v}^{\prime} & =0
\end{align*}
$$

where

$$
R=\frac{1}{\mu} \frac{\ell^{2}}{\tau}=\frac{\rho}{\bar{\mu}} \frac{\ell^{2}}{\tau}
$$

Here $\Delta^{\prime}, \nabla^{\prime}$ and $\operatorname{div}^{\prime}$ denote the analogous differential operations with respect to the variables $x^{\prime}$, and $G^{\prime}$ is the dimensionless description of $G$. The number $R$ is called Reynolds number. From the dimensions of $\mu$ it follows that $R$ is dimensionless.

Two motions are similar if they take place in homothetic domains with the same Reynolds number. Roughly speaking, the two domains have the same geometry and are mutually rescaled by a given length scale. The length scale being fixed then one rescales the time to obtain the same Reynolds number. For example, in building a vessel one is interested in investigating apriori how ${ }^{3}$ the shape of the hollow impacts on the motion of the surrounding fluid. One builds a model vessel, to be used in a limited laboratory environment, of the same shape but of reduced size, by rescaling the geometry by a fixed length. Experiments are performed with such a model in the same fluid where the vessel is intended to operate, so that the two fluids have the same viscosity. Finally having fixed the length scale one introduces a new time scale so that the Reynolds number remains the same. The two motions are then similar, and experimental laboratory operations correspond to those of the real fluid up to inverse length and time scales.

[^2]These remarks imply that the mathematical investigation of motions modeled by the Navier-Strokes equations reduces to investigate (7.3) with $R=1$, since space and time scales can always be chosen so that $R=1$. Denoting again by $\mathbf{v}, \mathbf{p}$, and $\mathbf{f}$ the indicated dimensionless quantities, and by $\Delta, \nabla$, div the homologous operations with respect to the indicated, rescaled, dimensionless variables, the mathematical problem consists in finding a velocity field $\mathbf{v}$ defined in $G \times \mathbb{R}^{+}$, such that

$$
\begin{align*}
\mathbf{v}_{t}-\Delta \mathbf{v}+(\mathbf{v} \cdot \nabla) \mathbf{v}+\nabla p=\mathbf{f} & \text { in } G \times \mathbb{R}^{+}  \tag{7.4}\\
\operatorname{div} \mathbf{v}=0 & \\
\mathbf{v}(\cdot, 0)=\mathbf{v}_{o} & \text { in } G \text { for } t=0
\end{align*}
$$

Here $\mathbf{v}_{o}$ is the initial velocity field defined in $G$ and assumed to be known. The determination of $\mathbf{v}$ hinges upon further information on its behavior on the boundary of $G$. For example, since the fluid is viscous it adheres to the boundary of its container $G$, so that $\mathbf{v}=0$ on $\partial G$. This is a Dirichlet datum of $\mathbf{v}$ on $\partial G$. On the other hand, the container might be impermeable so that no fluid outflows it at $\partial G$, that is $\mathbf{v} \cdot \mathbf{n}=0$ on $\partial G$ for all times. This is a Neumann datum of $\mathbf{v}$ on $\partial G$. These boundary information need not be homogeneous or could be intertwined, so that for example a Dirichlet datum is given on a portion $\partial_{1} G$ of $\partial G$ and a Neumann datum is prescribed on the remaining portion $\partial_{2} G=\partial G-\partial_{1} G$. Another kind of boundary condition will be given in the introductory section of the next chapter. While the physical formulation is simple, the corresponding mathematical problems are still not well understood and are the object of current investigations, and will be dealt with in the next Chapter.

## 8 Friction Tensor for Newtonian Viscous Fluids

### 8.1 Isotropic Tensors of the Fourth Order

Given a continuously differentiable velocity field $\mathbf{v}$ defined in $\mathbb{R}^{3}$, consider the expression

$$
\begin{equation*}
T_{i j}=\gamma_{i j h k} v_{h k}, \quad \text { where } \quad v_{h k}=\frac{\partial v_{h}}{\partial x_{k}}, \quad i, j, h, k=1,2,3 \tag{8.1}
\end{equation*}
$$

The nine numbers $T_{i j}$ are the representative entries of a tensor $\mathbb{T}$ of order 2 , with respect to a Cartesian triad $\Sigma$. Similarly $\left(\gamma_{i j h k}\right)$ is the $\Sigma$-representative of a 4 th-order tensor $\Gamma$. Let now $\Sigma^{\prime}$ be a new Cartesian triad obtained from $\Sigma$ by a rotation, realized by a unitary matrix $A: \Sigma \rightarrow \Sigma^{\prime}$. The vector field $x \rightarrow \mathbf{v}(x)$ is transformed into

$$
\begin{equation*}
\Sigma^{\prime} \ni y \longrightarrow \mathbf{v}^{\prime}(y)=A \mathbf{v}\left(A^{-1} y\right) \tag{8.2}
\end{equation*}
$$

and the representation of $\mathbb{T}$ in $\Sigma^{\prime}$ is

$$
T_{\ell m}^{\prime}=\gamma_{\ell m r s} v_{r s}^{\prime}, \quad \text { where } v_{r s}^{\prime}=\frac{\partial v_{r}^{\prime}}{\partial \xi_{s}}, \quad \ell, m, r, s=1,2,3
$$

Using (8.2) compute

$$
v_{r s}^{\prime}=A_{r h} A_{s k} \frac{\partial v_{h}}{\partial x_{k}}=A_{r h} A_{s k} v_{h k}
$$

Therefore,

$$
T_{\ell m}^{\prime}=A_{r h} A_{s k} \gamma_{\ell m r s} v_{h k}
$$

The tensor $\mathbb{T}$ is isotropic if its action on vectors is independent of the reference Cartesian triad, that is, if for all $\mathbf{w} \in \Sigma$

$$
\left(T_{i j}\right) \mathbf{w}=A^{-1}\left(T_{i j}^{\prime}\right) A \mathbf{w}, \quad \forall \mathbf{w} \in \Sigma
$$

Since $\mathbf{w} \in \Sigma$ is arbitrary

$$
T_{i j}=A_{\ell i} T_{\ell m}^{\prime} A_{m j}, \quad i, j=1,2,3
$$

Using these representations, it follows that $\mathbb{T}$ is isotropic if

$$
\gamma_{i j h k} v_{h k}=A_{\ell i} A_{m j} A_{r h} A_{s k} \gamma_{\ell m r s} v_{h k}, \quad i, j=1,2,3
$$

for all unitary matrices $A$. This in turn implies

$$
\begin{equation*}
\gamma_{i j h k}=A_{\ell i} A_{m j} A_{r h} A_{s k} \gamma_{\ell m r s}, \quad i, j, h, k=1,2,3 \tag{8.3}
\end{equation*}
$$

This is the condition for a 4th-order tensor $\Gamma$ to be isotropic.
Proposition 8.1 Let $\Gamma$ be a 4 th-order isotropic tensor. Then its representation with respect to a Cartesian triad $\Sigma$ is

$$
\begin{equation*}
\gamma_{i j h k}=\lambda \delta_{i j} \delta_{h k}+\mu_{1} \delta_{i h} \delta_{j k}+\mu_{2} \delta_{i k} \delta_{j h} \tag{8.4}
\end{equation*}
$$

where the constants $\lambda, \mu_{1}$ and $\mu_{2}$ are independent of $\Sigma$.
Proof. In (8.3) take the rotation matrix

$$
A=\left(\begin{array}{ccc}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{array}\right)
$$

For such a choice, the entries in (8.3) are non-zero only if the quadruple $i j h k$ coincides with $\ell m r s$, and in such a case

$$
\gamma_{i j h k}=A_{i i} A_{j j} A_{h h} A_{k k} \gamma_{i j h k}
$$

From the structure of the matrix $A$ above, one verifies that if in the quadruple $i j h k$, the index 3 occurs an odd number of times, then $\gamma_{i j h k}=-\gamma_{i j h k}$. Therefore,

$$
\begin{array}{ll}
\gamma_{i j h k}=0 & \begin{array}{l}
\text { if in the quadruple } i j h k \text { the index } 3 \\
\text { occurs an odd number of times. }
\end{array}
\end{array}
$$

Repeating the same arguments, for the choices of rotation matrices

$$
A=\left(\begin{array}{ccc}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{array}\right), \quad A=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{array}\right)
$$

one concludes that

$$
\begin{aligned}
& \gamma_{i j h k}=0
\end{aligned} \begin{aligned}
& \text { if in the quadruple } i j h k \text { anyone of the indices } \\
& 1,2,3 \text { occurs an odd number of times. }
\end{aligned}
$$

Therefore, the only non-zero elements are of the form

$$
\gamma_{i i h h}, \quad \gamma_{i h i h}, \quad \gamma_{i h h i}
$$

where repeated indices are not meant to be added. From (8.3) compute

$$
\begin{aligned}
& \gamma_{1133}=A_{\ell 1} A_{m 1} A_{r 3} A_{s 3} \gamma_{\ell m r s}=\delta_{\ell m} \delta_{r s} \gamma_{\ell m r s}, \\
& \gamma_{2233}=A_{\ell 2} A_{m 2} A_{r 3} A_{s 3} \gamma_{\ell m r s}=\delta_{\ell m} \delta_{r s} \gamma_{\ell m r s} .
\end{aligned}
$$

Therefore, $\gamma_{1133}=\gamma_{2233}$ and by symmetry

$$
\gamma_{1122}=\gamma_{1133}=\gamma_{2233}=\gamma_{3311}=\gamma_{2211}=\lambda
$$

If on the other hand all indices are equal, (8.3) gives the identity

$$
\gamma_{i i i i}=A_{\ell i} A_{m i} A_{r i} A_{s i} \gamma_{\ell m r s}=\delta_{\ell m r s} \gamma_{\ell m r s}=\gamma_{\ell \ell \ell \ell}, \quad i, \ell=1,2,3 .
$$

Analogous considerations for the remaining terms imply that there exist constants $\lambda, \mu_{1}, \mu_{2}, \theta$ such that

$$
\underbrace{\gamma_{i i h h}=\lambda, \quad \gamma_{i h i h}=\mu_{1}, \quad \gamma_{i h h i}=\mu_{2}}_{i \neq h}, \quad \gamma_{i i i i}=\theta, \quad \text { for all } i, h=1,2,3 .
$$

Putting this in (8.3) gives

$$
\begin{aligned}
\gamma_{i j h k}= & \sum_{\substack{\text { indices of the form } \\
i h h, i \neq h}} A_{\ell i} A_{m j} A_{r h} A_{s k} \gamma_{\ell m r s} \\
& +\sum_{\substack{\text { indices of the form } \\
\text { ihih, } i \neq h}} A_{\ell i} A_{m j} A_{r h} A_{s k} \gamma_{\ell m r s} \\
& +\sum_{\substack{\text { indices of the form } \\
i h h i, i \neq h}} A_{\ell i} A_{m j} A_{r h} A_{s k} \gamma_{\ell m r s}
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{\substack{\text { indices of the form } \\
i i i i}} A_{\ell i} A_{m j} A_{r h} A_{s k} \gamma_{\ell m r s} \\
& =\underbrace{\lambda \delta_{i j} \delta_{h k}+\mu_{1} \delta_{i h} \delta_{j k}+\mu_{2} \delta_{i k} \delta_{j h}}_{\substack{i \neq h}}+\theta \delta_{i j} \delta_{i k} \\
& =\lambda \delta_{i j} \delta_{h k}+\mu_{1} \delta_{i h} \delta_{j k}+\mu_{2} \delta_{i k} \delta_{j h}+\left[\theta-\left(\lambda+\mu_{1}+\mu_{2}\right)\right] \delta_{i j h k} .
\end{aligned}
$$

To conclude the proof it will be shown that this form of the tensor $\left(\gamma_{i j h k}\right)$ satisfies (8.3) for all unitary matrix $A$ if and only if $\theta=\lambda+\mu_{1}+\mu_{2}$. Indeed, from (8.3)

$$
\begin{aligned}
& \lambda \delta_{i j} \delta_{h k}+\mu_{1} \delta_{i h} \delta_{j k}+\mu_{2} \delta_{i k} \delta_{j h}+\left[\theta-\left(\lambda+\mu_{1}+\mu_{2}\right)\right] \delta_{i j h k} \\
= & A_{\ell i} A_{m j} A_{r h} A_{s k}\left\{\lambda \delta_{\ell m} \delta_{r s}+\mu_{1} \delta_{\ell r} \delta_{m s}+\mu_{2} \delta_{\ell s} \delta_{m r}\right. \\
& \left.+\left[\theta-\left(\lambda+\mu_{1}+\mu_{2}\right)\right] \delta_{i j h k}\right\} \\
= & \lambda \delta_{i j} \delta_{h k}+\mu_{1} \delta_{i h} \delta_{j k}+\mu_{2} \delta_{i k} \delta_{j h}+A_{\ell i} A_{m j} A_{r h} A_{s k}\left[\theta-\left(\lambda+\mu_{1}+\mu_{2}\right)\right] \delta_{\ell m r s}
\end{aligned}
$$

Therefore, the tensor on the left-hand side satisfies (8.3) for all unitary matrices $A$ if

$$
\left[\theta-\left(\lambda+\mu_{1}+\mu_{2}\right)\right]\left(\delta_{i j h k}-A_{\ell i} A_{m j} A_{r h} A_{s k} \delta_{\ell m r s}\right)=0
$$

for all unitary matrices $A$. This is possible only if the coefficient independent of the indices is zero.

## ANALYSIS OF THE NAVIER-STOKES EQUATIONS

## 1 Navier-Stokes Equations in Dimensionless Form

Let $E$ be a physical open set in $\mathbb{R}^{3}$ filled with a fluid of dynamic viscosity $\mu$ and constant density $\rho$, whose infinitesimal ideal particles at $x \in E$ at time $t$ move with velocity $\mathbf{v}=\left(v_{1}, v_{2}, v_{3}\right)$ function of $(x, t)$, and are acted upon by the pressure $(x, t) \rightarrow p(x, t)$, and by possible external force densities $\mathbf{f}_{e}(x, t)$, per unit volume. Enforcing the local, pointwise conservation of momentum along each of the ideal Lagrangian paths $t \rightarrow x(t)$, yields the Navier-Stokes system,

$$
\begin{align*}
\rho\left[\mathbf{v}_{t}+(\mathbf{v} \cdot \nabla) \mathbf{v}\right]-\mu \Delta_{x} \mathbf{v}+\nabla p & =\mathbf{f}_{e}  \tag{1.1}\\
\operatorname{div}_{x} \mathbf{v} & =0
\end{align*} \quad \text { in } E \times(0, \infty) .
$$

Here $\Delta_{x}, \nabla$ and $\operatorname{div}_{x}$ denote the corresponding differential operation with respect to the physical space variables $x$. If $\mathbf{f}_{e}$ is conservative, such as for example gravity, then $\mathbf{f}_{e}=\nabla F$ for some given potential $F$. In such a case (1.1) can be written in homogeneous form by redefining $p$ as $(p-F)$.

The various terms in (1.1) are written in terms of pre-chosen physical unit length $[L]$ and time $[T]$ and corresponding unit velocity $[V]=[L][T]^{-1}$, unit pressure $[P]=\rho[V]^{2}$, unit force density $[F]=\rho[V][T]^{-1}$ and unit dynamic viscosity $[\mu]=\rho[V][L]$. They can be written in dimensionless form by introducing dimensionless space variables $y=x[L]^{-1}$ and time $\tau=t[T]^{-1}$ and corresponding dimensionless velocities, pressures and force densities

$$
\tilde{\mathbf{v}}(y, \tau)=\frac{\mathbf{v}(y[L], \tau[T])}{[V]}, \quad \tilde{p}(y, \tau)=\frac{p(y[L], \tau[T])}{[P]}, \quad \tilde{\mathbf{f}}=\frac{\mathbf{f}_{e}(y[L], \tau[T])}{[F]}
$$

Denote by $\tilde{E}$ the rescaled physical domain $E$ expressed in terms of dimensionless coordinates. Then, dividing (1.1) by $\rho$ and formally by $[V][T]^{-1}$, yields

$$
\begin{align*}
\tilde{\mathbf{v}}_{\tau}-\frac{1}{\mathcal{R} e} \Delta_{y} \tilde{\mathbf{v}}+\left(\tilde{\mathbf{v}} \cdot \nabla_{y}\right) \tilde{\mathbf{v}}+\nabla_{y} \tilde{p} & =\tilde{f} \quad \text { in } \tilde{E}  \tag{1.2}\\
\operatorname{div}_{y} \tilde{\mathbf{v}} & =0
\end{align*}
$$

where $\mathcal{R} e$ is the Reynolds number ${ }^{1}$ of the system corresponding to the units $[L]$ and $[T]$ and defined by

$$
\mathcal{R} e \stackrel{\text { def }}{=} \frac{\rho[V][L]}{\mu}
$$

While $\mathcal{R} e$ is dimensionless, its numerical value depends on the choice of $[L]$ and $[T]$. Indeed the dynamic viscosity $\mu$ for a fluid of density $\rho$, is experimentally determined in terms of some given units, say for example $\mathrm{cm}^{2} \mathrm{sec}^{-1}$. Expressing them in terms of new units $[L]$ and $[T]$, changes the numerical value of $\mathcal{R} e$. The coefficient of $\Delta_{y}$ in (1.2) is the dimensionless kinematic viscosity $\nu$ of the rescaled fluid.

This rescaling procedure is at the basis of predicting experimentally non accessible fluid flows in large scale domains, such as air past an airfoil or water past a vessel. The physical domains are rescaled to experimentally accessible dimensions, such as laboratory water channels or wind tunnels, with properly redefined Reynolds number. Information provided by the dimensionless system (1.2) is then rescaled back to the physical domain.

To simplify the symbolism we continue to denote by $x, t, \mathbf{v}, p$ and $\mathbf{f}$ the rescaled, dimensionless quantities and rewrite the Navier-Stokes system (1.1) in the dimensionless domain $E$, for dimensionless times $t>0$ in the form

$$
\begin{align*}
\mathbf{v}_{t}-\nu \Delta \mathbf{v}+(\mathbf{v} \cdot \nabla) \mathbf{v}+\nabla p & =\mathbf{f} \quad \text { in } E \times(0, \infty)  \tag{1.3}\\
\operatorname{div} \mathbf{v} & =0
\end{align*} \quad \text {, }
$$

with $\nu=\mathcal{R} e^{-1}$. Typically one prescribes the velocity field $\mathbf{v}_{o}=\mathbf{v}(\cdot, 0)$ at time $t=0$ and $\mathbf{v}(\cdot, t)=\mathbf{g}(\cdot, t)$ on $\partial E$ for $t>0$ and seeks to solve (1.3) subject to these data.

If $E$ is a rigid container at rest with respect to an inertial system, then $\partial E$ acts as a rigid wall and $\mathbf{g}=0$, by viscosity. This is the so-called no-slip condition. The case $\mathbf{g} \neq 0$ may occur when $\partial E$ is itself in motion with respect to an inertial system. In the applications, other types of boundary conditions have been considered. We talk of kinematic condition, when the normal component of the velocity vanishes at the boundary, that is, the velocity $\mathbf{v}$ is tangent to the boundary:

$$
\mathbf{v}(\cdot, t) \cdot \mathbf{n}=0 \quad \text { on } \quad \partial E
$$

for $t>0$, where $\mathbf{n}$ is the outward unit normal to the boundary $\partial E$. In 1823 Navier proposed a more general condition, namely the so-called Navier boundary condition, which, roughly speaking, states that the tangential component of the velocity is proportional to the tangential stress at the boundary. We will not consider these different boundary conditions in the following.

The system (1.3) is formal since, even by prescribing smooth initial and boundary data $\mathbf{v}_{o}$ and $\mathbf{g}$ and forcing term $\mathbf{f}$, one cannot apriori guarantee that $\mathbf{v}$ and $p$ are so regular as to give pointwise meaning to its various terms.

[^3]
## 2 Steady State Flow with Homogeneous Boundary Data

Let $E$ be a bounded domain in $\mathbb{R}^{3}$ with boundary $\partial E$, and consider, formally, the steady-state flow in $E$,

$$
\begin{align*}
-\nu \Delta \mathbf{v}+(\mathbf{v} \cdot \nabla) \mathbf{v}+\nabla p & =\mathbf{f}, \\
\operatorname{div} \mathbf{v} & =0, \quad \text { in } E .  \tag{2.1}\\
\left.\mathbf{v}\right|_{\partial E} & =0
\end{align*}
$$

Introduce the space of functions

$$
\begin{align*}
\mathcal{V} & =\left\{\varphi \in C_{o}^{\infty}\left(E ; \mathbb{R}^{3}\right) \text { such that } \operatorname{div} \varphi=0 \text { in } E\right\} \\
H & =\left\{\text { closure of } \mathcal{V} \text { in the norm of } L^{2}\left(E ; \mathbb{R}^{3}\right)\right\}  \tag{2.2}\\
V & =\left\{\text { closure of } \mathcal{V} \text { in the norm of } W_{o}^{1,2}\left(E ; \mathbb{R}^{3}\right)\right\}
\end{align*}
$$

Formally inner-multiply the first of (2.1) by $\varphi \in \mathcal{V}$ and integrate by parts in $E$. Since $\mathbf{v}$ and $\varphi$ are both divergence free, obtain formally

$$
\begin{equation*}
\int_{E}[\nu \nabla \mathbf{v}: \nabla \boldsymbol{\varphi}-\mathbf{v} \cdot(\mathbf{v} \cdot \nabla) \boldsymbol{\varphi}-\mathbf{f} \cdot \boldsymbol{\varphi}] d x=0 \tag{2.3}
\end{equation*}
$$

where

$$
\nabla \mathbf{v}: \nabla \boldsymbol{\varphi}=\sum_{j=1}^{3} \nabla v_{j} \cdot \nabla \varphi_{j}
$$

Here we have used the relation

$$
\int_{E} \boldsymbol{\psi} \cdot(\boldsymbol{\zeta} \cdot \nabla) \boldsymbol{\varphi} d x=-\int_{E} \boldsymbol{\varphi} \cdot(\boldsymbol{\zeta} \cdot \nabla) \boldsymbol{\psi} d x
$$

valid for any triple of solenoidal vectors $\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\zeta} \in W^{1,2}\left(E ; \mathbb{R}^{3}\right)$ such that at least one of them is in $V$. As a consequence

$$
\int_{E} \varphi \cdot(\boldsymbol{\zeta} \cdot \nabla) \varphi d x=0
$$

These calculus operations will be repeatedly used without specific mention.
By the Sobolev embedding Theorem applied with $N=3$ and $p=2$, there exists a constant $\gamma$ independent of $E$ and $\mathbf{v}$, such that

$$
\begin{equation*}
\|\mathbf{v}\|_{6} \leq \gamma\|\nabla \mathbf{v}\|_{2} \quad \text { for all } \mathbf{v} \in V \tag{2.4}
\end{equation*}
$$

Therefore, for all such $\mathbf{v}$

$$
\begin{equation*}
\|\mathbf{v}\|_{2} \leq \gamma|E|^{\frac{1}{3}}\|\nabla \mathbf{v}\|_{2} \quad \text { and } \quad\|\mathbf{v}\|_{4} \leq \gamma|E|^{\frac{1}{12}}\|\nabla \mathbf{v}\|_{2} \tag{2.5}
\end{equation*}
$$

As a consequence

$$
\begin{equation*}
\gamma_{o}\|\mathbf{v}\|_{V} \leq\|\nabla \mathbf{v}\|_{2} \leq\|\mathbf{v}\|_{V} \quad \text { where } \gamma_{o}=\frac{1}{\gamma|E|^{\frac{1}{3}}+1} \tag{2.6}
\end{equation*}
$$

for all $\mathbf{v} \in V$, where the rigorous definition of norm in $V$ is given in (3.1) below. By these inequalities, all terms in (2.3) are well defined for all $\varphi \in V$ and $\mathbf{f} \in L^{\frac{6}{5}}\left(E ; \mathbb{R}^{3}\right)$. Thus, having prescribed one such $\mathbf{f}$, we define a weak solution of (2.1) as a function $\mathbf{v} \in V$ satisfying (2.3) for all $\varphi \in V$. The homogeneous boundary data on $\partial E$ are taken in the sense of the membership $\mathbf{v} \in V$. The same membership guarantees that $\operatorname{div} \mathbf{v}=0$ in the weak form

$$
\int_{E} \mathbf{v} \cdot \nabla \varphi d x=0 \quad \text { for all } \varphi \in C^{\infty}(E)
$$

By this definition of solution, the choice $\varphi=\mathbf{v}$ is admissible in (2.3) yielding the basic energy estimate

$$
\begin{equation*}
\nu\|\nabla \mathbf{v}\|_{2} \leq \gamma\|\mathbf{f}\|_{\frac{6}{5}} \tag{2.7}
\end{equation*}
$$

to be satisfied by any weak solution to (2.1), where $\gamma$ is the constant of the embedding of $V$ into $L^{6}\left(E ; \mathbb{R}^{3}\right)$. Thus if $\mathbf{f}=0$ then $\mathbf{v}=0$ is the only weak solution of (2.1).

### 2.1 Uniqueness of Solutions to (2.1)

Let $\mathbf{v}_{1}$ and $\mathbf{v}_{2}$ be weak solutions to (2.1) corresponding to the choice of $\mathbf{f} \in$ $L^{\frac{6}{5}}\left(E ; \mathbb{R}^{3}\right)$. Write (2.3) for $\mathbf{v}_{1}$ and $\mathbf{v}_{2}$, subtract the expression so obtained and in the resulting integral identity choose $\varphi=\mathbf{w} \stackrel{\text { def }}{=}\left(\mathbf{v}_{1}-\mathbf{v}_{2}\right)$, to obtain

$$
\begin{aligned}
\nu\|\nabla \mathbf{w}\|_{2}^{2} & =\int_{E}\left[\mathbf{v}_{1} \cdot(\mathbf{w} \cdot \nabla) \mathbf{w}+\mathbf{w} \cdot\left(\mathbf{v}_{2} \cdot \nabla\right) \mathbf{w}\right] d x \\
& \leq\left(\left\|\mathbf{v}_{1}\right\|_{4}+\left\|\mathbf{v}_{2}\right\|_{4}\right)\|\mathbf{w}\|_{4}\|\nabla \mathbf{w}\|_{2}
\end{aligned}
$$

Since $\mathbf{v}_{j}$ are solutions, combining (2.5) and (2.7) gives

$$
\|\mathbf{w}\|_{4} \leq \gamma|E|^{\frac{1}{12}}\|\nabla \mathbf{w}\|_{2} \quad \text { and } \quad\left\|\mathbf{v}_{j}\right\|_{4} \leq \frac{\gamma^{2}}{\nu}|E|^{\frac{1}{3}}\|\mathbf{f}\|_{\frac{6}{5}}
$$

Therefore,

$$
\|\nabla \mathbf{w}\|_{2} \leq 2 \gamma\left(\frac{\gamma}{\nu}\right)^{2}|E|^{\frac{5}{12}}\|\mathbf{f}\|_{\frac{6}{5}}\|\nabla \mathbf{w}\|_{2}
$$

If the coefficient on the right-hand side is less than 1 , then $\mathbf{w}=0$ and the solution is unique. Such a coefficient depends on the absolute constant $\gamma$ of the embedding $V \subset L^{6}\left(E ; \mathbb{R}^{3}\right)$, on the size of $E$, the viscosity $\nu$, and the nature of the forcing term $\mathbf{f}$. Given $E$ and $\mathbf{f}$ uniqueness holds if the Reynolds number of the system is sufficiently small or equivalently if the fluid is sufficiently viscous.

It should be noted that the definition of weak solution does not depend on the pressure $p$, which itself is an unknown to be found from (2.1).

## 3 Existence of Solutions to (2.1)

The spaces $H$ and $V$ introduced in (2.2) are separable Hilbert spaces by the inner products

$$
\begin{align*}
& H \ni(\mathbf{u}, \mathbf{v}) \rightarrow\langle\mathbf{u}, \mathbf{v}\rangle_{H} \stackrel{\text { def }}{=} \int_{E} \mathbf{u} \cdot \mathbf{v} d x \\
& V \ni(\mathbf{u}, \mathbf{v}) \rightarrow(\mathbf{u}, \mathbf{v})_{V} \stackrel{\text { def }}{=}\langle\mathbf{u}, \mathbf{v}\rangle_{H}+\int_{E} \nabla \mathbf{u}: \nabla \mathbf{v} d x \tag{3.1}
\end{align*}
$$

By (2.6) the inner product $(\cdot, \cdot)_{V}$ is equivalent to

$$
V \ni(\mathbf{u}, \mathbf{v}) \rightarrow\langle\mathbf{u}, \mathbf{v}\rangle_{V}=\int_{E} \nabla \mathbf{u}: \nabla \mathbf{v} d x=\sum_{j=1}^{3}\left\langle\nabla u_{j}, \nabla v_{j}\right\rangle_{H}
$$

which from now on we adopt. Having fixed $\mathbf{f} \in L^{\frac{6}{5}}\left(E ; \mathbb{R}^{3}\right)$ and $\mathbf{v} \in V$, return to (2.3) and consider the two linear maps

$$
\begin{aligned}
& V \ni \boldsymbol{\varphi} \rightarrow \stackrel{\text { def }}{=} \int_{E} \mathbf{f} \cdot \boldsymbol{\varphi} d x \\
& V \ni \boldsymbol{\varphi} \rightarrow \stackrel{\text { def }}{=} \int_{E} \mathbf{v} \cdot(\mathbf{v} \cdot \nabla) \boldsymbol{\varphi} d x .
\end{aligned}
$$

By Hölder's inequality and the embedding $V \subset L^{6}\left(E ; \mathbb{R}^{3}\right)$

$$
\left|\int_{E} \mathbf{f} \cdot \varphi d x\right| \leq \gamma\|\mathbf{f}\|_{\frac{6}{5}}\|\nabla \varphi\|_{2}
$$

Therefore, the first is a bounded linear functional on $V$. By the Riesz representation theorem, there exists a unique $\mathbf{F} \in V$ such that ${ }^{2}$

$$
V \ni \varphi \rightarrow \int_{E} \mathbf{f} \cdot \varphi d x=\langle\mathbf{F}, \varphi\rangle_{V}
$$

Likewise, by the same embedding and (2.5)

$$
\left|\int_{E} \mathbf{v} \cdot(\mathbf{v} \cdot \nabla) \varphi d x\right| \leq \gamma^{2}|E|^{\frac{1}{6}}\|\nabla \mathbf{v}\|_{2}^{2}\|\nabla \varphi\|_{2}
$$

Therefore, also the second map, for every fixed $\mathbf{v} \in V$, is a bounded linear functional in $V$. By the Riesz representation theorem, there exists a unique $B(\mathbf{v}) \in V$ such that

$$
V \ni \varphi \rightarrow \int_{E} \mathbf{v} \cdot(\mathbf{v} \cdot \nabla) \varphi d x=\langle B(\mathbf{v}), \varphi\rangle_{V}
$$

With these identifications, the weak formulation (2.3) can be recast in the form

[^4]$$
V \ni \boldsymbol{\varphi} \rightarrow \nu\langle\mathbf{v}, \boldsymbol{\varphi}\rangle_{V}=\langle B(\mathbf{v})+\mathbf{F}, \boldsymbol{\varphi}\rangle_{V} .
$$

Equivalently, in functional form

$$
\begin{equation*}
\mathbf{v}=\mathcal{B}(\mathbf{v}) \quad \text { in } \quad V^{*} \quad \text { where } \quad \mathcal{B}(\mathbf{v})=\frac{1}{\nu}(B(\mathbf{v})+\mathbf{F}) \tag{3.2}
\end{equation*}
$$

and $V^{*}$ denotes the dual of $V$ identified with $V$ itself up to an isometric isomorphism. Thus, existence of weak solution to (2.1) in the sense of (2.3) is equivalent to finding a fixed point of the map $V \ni \mathbf{v} \rightarrow \mathcal{B}(\mathbf{v}) \in V^{*}$.

Lemma 3.1 The map $\mathcal{B}(\cdot): V \rightarrow V^{*}$ is compact.
Proof. Since $V$ and $V^{*}$ are separable metric spaces, compactness is equivalent to sequential compactness. Let $K$ be a bounded subset of $V$, i.e., there exists a constant $C$ such that $\|\mathbf{v}\|_{V} \leq C$ for all $\mathbf{v} \in K$. The image $\mathcal{B}(K)$ is pre-compact in $V^{*}$ if for every sequence $\left\{\mathbf{v}_{n}\right\} \subset K$ there exists a subsequence $\left\{\mathbf{v}_{n^{\prime}}\right\} \subset\left\{\mathbf{v}_{n}\right\}$ such that $\left\{\mathcal{B}\left(\mathbf{v}_{n^{\prime}}\right)\right\}$ is a Cauchy sequence in the operator topology of $V^{*}$. By the Rellich-Kondrachov compact embedding theorem, the embedding $V \supset$ $K \hookrightarrow L^{p}\left(E ; \mathbb{R}^{3}\right)$ is compact for all $1 \leq p<6$. Therefore, having fixed $1 \leq p<$ 6 , from every sequence $\left\{\mathbf{v}_{n}\right\} \subset K$ one can extract a subsequence $\left\{\mathbf{v}_{n^{\prime}}\right\} \subset\left\{\mathbf{v}_{n}\right\}$ which is Cauchy in the topology of $L^{p}\left(E ; \mathbb{R}^{3}\right)$. Hence, to show $\mathcal{B}(K)$ is precompact in $V^{*}$ it suffices to show that for every sequence $\left\{\mathbf{v}_{n}\right\} \subset K$, Cauchy in $L^{4}\left(E ; \mathbb{R}^{3}\right)$ the corresponding sequence $\left\{\mathcal{B}\left(\mathbf{v}_{n}\right)\right\}$ is Cauchy in the operator topology of $V^{*}$. Having fixed one such sequence $\left\{\mathbf{v}_{n}\right\} \subset K$, the action of $\nu\left[\mathcal{B}\left(\mathbf{v}_{n}\right)-\mathcal{B}\left(\mathbf{v}_{m}\right)\right]$ on elements $\varphi \in V$, is computed from

$$
\begin{aligned}
\left\langle\nu\left[\mathcal{B}\left(\mathbf{v}_{n}\right)-\mathcal{B}\left(\mathbf{v}_{m}\right)\right], \boldsymbol{\varphi}\right\rangle_{V}= & \int_{E}\left[\mathbf{v}_{n} \cdot\left(\mathbf{v}_{n} \cdot \nabla\right)-\mathbf{v}_{m} \cdot\left(\mathbf{v}_{m} \cdot \nabla\right)\right] \boldsymbol{\varphi} d x \\
= & \int_{E}\left(\mathbf{v}_{n}-\mathbf{v}_{m}\right) \cdot\left(\mathbf{v}_{n} \cdot \nabla\right) \varphi d x \\
& +\int_{E} \mathbf{v}_{m} \cdot\left(\left(\mathbf{v}_{n}-\mathbf{v}_{m}\right) \cdot \nabla\right) \boldsymbol{\varphi} d x \\
\leq & \left(\left\|\mathbf{v}_{n}\right\|_{4}+\left\|\mathbf{v}_{m}\right\|_{4}\right)\left\|\mathbf{v}_{n}-\mathbf{v}_{m}\right\|_{4}\|\boldsymbol{\varphi}\|_{V} \\
\leq & \gamma|E|^{\frac{1}{12}}\left(\left\|\mathbf{v}_{n}\right\|_{V}+\left\|\mathbf{v}_{m}\right\|_{V}\right)\left\|\mathbf{v}_{n}-\mathbf{v}_{m}\right\|_{4}\|\boldsymbol{\varphi}\|_{V} .
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left\|\mathcal{B}\left(\mathbf{v}_{n}\right)-\mathcal{B}\left(\mathbf{v}_{m}\right)\right\|_{V^{*}} & =\sup _{\|\boldsymbol{\varphi}\|_{V}=1}\left\langle\left[\mathcal{B}\left(\mathbf{v}_{n}\right)-\mathcal{B}\left(\mathbf{v}_{m}\right)\right], \boldsymbol{\varphi}\right\rangle_{V} \\
& \leq 2 C \gamma|E|^{\frac{1}{12}}\left\|\mathbf{v}_{n}-\mathbf{v}_{m}\right\|_{4}
\end{aligned}
$$

Consider next the family of variants of (3.2)

$$
\begin{equation*}
\mathbf{v}=\lambda \mathcal{B}(\mathbf{v}) \quad \text { in } V \quad \text { for } \quad \lambda \in(0,1) \tag{3.3}
\end{equation*}
$$

If $\mathbf{v}_{\lambda}$ is a solution of (3.3), it is also a solution of (2.3) with $\nu$ replaced by $\nu / \lambda$. As such, the apriori estimates (2.6) and (2.7) remain in force with $\mathbf{v}$ replaced by $\mathbf{v}_{\lambda}$ and $\nu$ replaced by $\nu / \lambda$, i.e.,

$$
\left\|\mathbf{v}_{\lambda}\right\|_{V} \leq \frac{1}{\gamma_{o}}\left\|\nabla \mathbf{v}_{\lambda}\right\|_{2} \leq \frac{\lambda}{\nu} \frac{\gamma}{\gamma_{o}}\|\mathbf{f}\|_{\frac{6}{5}} .
$$

Therefore, all possible solutions of (3.3) are uniformly bounded in $\lambda$. Existence of solutions of (3.2), and hence of (2.1), now follows from the Schauder-Leray Fixed Point Theorem.

Theorem 3.1 (Schauder-Leray [28]). Let $T$ be a continuous, compact mapping from a Banach space $\left\{X ;\|\cdot\|_{X}\right\}$ into itself, such that all possible solutions of $x=\lambda T(x)$ are equi-bounded uniformly in $\lambda \in(0,1)$. Then $T$ has a fixed point.

## 4 Non-Homogeneous Boundary Data

Let $E$ be a simply connected, bounded domain in $\mathbb{R}^{3}$ with boundary $\partial E$ of class $C^{1}$ and satisfying the segment property and consider, formally, the steady-state flow in $E$,

$$
\begin{align*}
-\nu \Delta \mathbf{v}+(\mathbf{v} \cdot \nabla) \mathbf{v}+\nabla p & =\mathbf{f}, \\
\operatorname{div} \mathbf{v} & =0, \quad \text { in } E  \tag{4.1}\\
\left.\mathbf{v}\right|_{\partial E} & =\mathbf{a}
\end{align*}
$$

where $\mathbf{a}$ is a vector valued function defined on $\partial E$, whose regularity will be specified as we proceed. If $\mathbf{v}$ is a solution of (4.1) then, formally, by Green's theorem,

$$
\begin{equation*}
0=\int_{E} \operatorname{div} \mathbf{v} d x=\int_{\partial E} \mathbf{a} \cdot \mathbf{n} d \sigma \tag{4.2}
\end{equation*}
$$

where $\mathbf{n}$ is the outward unit normal to $\partial E$. This is then a necessary condition to be imposed on a for the solvability of (4.1). Solvability of (4.1) hinges on extending $\mathbf{a}$ with a divergence free vector valued function $\mathbf{b}$ defined in $E$. The smoothness of $\mathbf{b}$ and the meaning of $\mathbf{b}=\mathbf{a}$ on $\partial E$ will be made precise as we proceed. Assuming that such an extension can be found, seek a solution to (4.1) in the form $\mathbf{v}=\mathbf{b}+\mathbf{u}$, where formally

$$
\begin{align*}
-\nu \Delta \mathbf{u}+(\mathbf{u} \cdot \nabla) \mathbf{u}+(\mathbf{b} \cdot \nabla) \mathbf{u}+(\mathbf{u} \cdot \nabla) \mathbf{b}+\nabla p & =\mathbf{g}, \\
\operatorname{div} \mathbf{u} & =0, \quad \text { in } E,  \tag{4.3}\\
\left.\mathbf{u}\right|_{\partial E} & =\mathbf{0}
\end{align*}
$$

and

$$
\begin{equation*}
\mathbf{g}=\mathbf{f}+\nu \Delta \mathbf{b}-(\mathbf{b} \cdot \nabla) \mathbf{b} \tag{4.4}
\end{equation*}
$$

Solutions of (4.3) are sought in $V$ with the equation being interpreted in its weak form

$$
\begin{align*}
\nu \int_{E} & \nabla \mathbf{u}: \nabla \varphi d x \\
& =\int_{E}\{[\mathbf{u} \cdot(\mathbf{u} \cdot \nabla)+\mathbf{u} \cdot(\mathbf{b} \cdot \nabla)+\mathbf{b} \cdot(\mathbf{u} \cdot \nabla)] \varphi+\mathbf{g} \cdot \varphi\} d x \tag{4.5}
\end{align*}
$$

for all $\varphi \in V$. Taking $\varphi=\mathbf{u}$ gives the apriori estimate

$$
\begin{equation*}
\left(\nu-\gamma|E|^{\frac{1}{12}}\|\mathbf{b}\|_{4}\right)\|\nabla \mathbf{u}\|_{2}^{2} \leq\left|\int_{E} \mathbf{g} \cdot \mathbf{u} d x\right| \tag{4.6}
\end{equation*}
$$

where $\gamma$ is the constant of the embedding of $V$ into $L^{6}\left(E ; \mathbb{R}^{3}\right)$. The right-hand is finite if $\mathbf{f} \in L^{\frac{6}{5}}\left(E ; \mathbb{R}^{3}\right)$ and $\mathbf{b} \in W^{1,2}\left(E ; \mathbb{R}^{3}\right)$, since by the Sobolev-Nikol'skii embedding theorem, this implies $\mathbf{b} \in L^{6}\left(E ; \mathbb{R}^{3}\right)$. Indeed,

$$
\begin{equation*}
\left|\int_{E} \mathbf{g} \cdot \mathbf{u} d x\right| \leq\left[\gamma\|\mathbf{f}\|_{\frac{6}{5}}+\nu\|\nabla \mathbf{b}\|_{2}+\|\mathbf{b}\|_{4}^{2}\right]\|\nabla \mathbf{u}\|_{2} \tag{4.7}
\end{equation*}
$$

where again $\gamma$ is the constant of the embedding of $V$ into $L^{6}\left(E ; \mathbb{R}^{3}\right)$.
If the domain $E$ has boundary $\partial E$ of class $C^{1}$ and satisfies in addition the segment property, functions $\mathbf{b} \in W^{1,2}\left(E ; \mathbb{R}^{3}\right)$ have traces on $\partial E$ in the fractional Sobolev space

$$
\begin{equation*}
\left.\mathbf{b}\right|_{\partial E}=\mathbf{a} \in W^{\frac{1}{2}, 2}\left(\partial E ; \mathbb{R}^{3}\right) \tag{4.8}
\end{equation*}
$$

Henceforth given a boundary datum $\mathbf{a} \in W^{\frac{1}{2}, 2}\left(\partial E ; \mathbb{R}^{3}\right)$, we assume it can be extended into a solenoidal vector field $\mathbf{b} \in W^{1,2}\left(E ; \mathbb{R}^{3}\right)$. A compatibility condition for such an extension to exist is that a has zero flux across $\partial E$ as indicated by (4.2). We also assume that such an extension can be constructed to satisfy

$$
\begin{equation*}
\gamma|E|^{\frac{1}{12}}\|\mathbf{b}\|_{4} \leq \frac{1}{2} \nu \tag{4.9}
\end{equation*}
$$

The actual construction of an extension $\mathbf{b}$ satisfying (4.8) is carried out in Section 4.2c of the Complements. Moreover, we assume that (4.9) can be derived from $(4.7 \mathrm{c})$. Accepting it for the moment, this last requirement combined with (4.6)-(4.7) yields the apriori estimate

$$
\begin{equation*}
\|\nabla \mathbf{u}\|_{2} \leq \frac{2 \gamma}{\nu}\left[\|\mathbf{f}\|_{\frac{6}{5}}+\nu\|\nabla \mathbf{b}\|_{2}+\|\mathbf{b}\|_{4}^{2}\right] \tag{4.10}
\end{equation*}
$$

to be satisfied by any weak solution to (4.1).

### 4.1 Uniqueness of Solutions to (4.1)

If $\mathbf{u}_{1}$ and $\mathbf{u}_{2}$ in $V$ solve (4.1) write their weak formulations (4.5), subtract them out and in the integral identity so obtained take the testing function $\boldsymbol{\varphi}=\left(\mathbf{u}_{1}-\mathbf{u}_{2}\right) \stackrel{\text { def }}{=} \mathbf{w}$, and make use of the embedding (2.5) and the upper bound (4.10) to be satisfied by all weak solutions to (4.1), to obtain

$$
\begin{aligned}
\|\nabla \mathbf{w}\|_{2}^{2} & =\frac{1}{\nu} \int_{E}\left[\mathbf{u}_{1} \cdot(\mathbf{w} \cdot \nabla) \mathbf{w}+\mathbf{w} \cdot\left(\mathbf{u}_{2} \cdot \nabla\right) \mathbf{w}+\mathbf{b} \cdot(\mathbf{w} \cdot \nabla) \mathbf{w}\right] d x \\
& \leq \frac{1}{\nu}\left(\left\|\mathbf{u}_{1}\right\|_{4}+\left\|\mathbf{u}_{2}\right\|_{4}+\|\mathbf{b}\|_{4}\right)\|\mathbf{w}\|_{4}\|\nabla \mathbf{w}\|_{2} \\
& \leq \frac{\gamma}{\nu}|E|^{\frac{1}{12}}\left(4 \frac{\gamma^{2}}{\nu}|E|^{\frac{1}{12}}\left[\|\mathbf{f}\|_{\frac{6}{5}}+\nu\|\nabla \mathbf{b}\|_{2}+\|\mathbf{b}\|_{4}^{2}\right]+\|\mathbf{b}\|_{4}\right)\|\nabla \mathbf{w}\|_{2}^{2}
\end{aligned}
$$

If the coefficient of $\|\nabla \mathbf{w}\|_{2}^{2}$ on the right-hand side does not exceed 1 then $\mathbf{w}=0$ and the problem admits at most one solution. The uniqueness condition hinges on several factors including $|E|$ and the size of the extension $\mathbf{b}$ through the norms $\|\nabla \mathbf{b}\|_{2}$ and $\|\mathbf{b}\|_{4}$. The key condition however is expressed by the smallness of the Reynolds number $\mathcal{R} e=\nu^{-1}$. Thus uniqueness holds if the Reynolds number is sufficiently small or equivalently if the fluid is sufficiently viscous.

### 4.2 Existence of Solutions to (4.1)

Consider the linear maps

$$
\begin{aligned}
& V \ni \varphi \rightarrow \stackrel{\text { def }}{=} \int_{E} \mathbf{g} \cdot \boldsymbol{\varphi} d x \\
& V \ni \varphi \rightarrow \stackrel{\text { def }}{=} \int_{E}[\mathbf{u} \cdot(\mathbf{u} \cdot \nabla)+\mathbf{u} \cdot(\mathbf{b} \cdot \nabla)+\mathbf{b} \cdot(\mathbf{u} \cdot \nabla)] \varphi d x
\end{aligned}
$$

Estimate

$$
\left|\int_{E} \mathbf{g} \cdot \boldsymbol{\varphi} d x\right| \leq \gamma\left[\|\mathbf{f}\|_{\frac{6}{5}}+\nu\|\nabla \mathbf{b}\|_{2}+\|\mathbf{b}\|_{4}^{2}\right]\|\nabla \boldsymbol{\varphi}\|_{2}
$$

Therefore, the first is a bounded linear functional in $V$. By the Riesz representation theorem there exists a unique $\mathbf{G} \in V$ such that

$$
V \ni \varphi \rightarrow \int_{E} \mathbf{g} \cdot \varphi d x=\langle\mathbf{G}, \varphi\rangle_{V}
$$

Likewise, estimate

$$
\begin{aligned}
& \left|\int_{E}[\mathbf{u} \cdot(\mathbf{u} \cdot \nabla)+\mathbf{u} \cdot(\mathbf{b} \cdot \nabla)+\mathbf{b} \cdot(\mathbf{u} \cdot \nabla)] \boldsymbol{\varphi} d x\right| \\
& \quad \leq \gamma|E|^{\frac{1}{12}}\left(\gamma|E|^{\frac{1}{12}}\|\nabla \mathbf{u}\|_{2}+2\|\mathbf{b}\|_{4}\right)\|\nabla \mathbf{u}\|_{2}\|\nabla \varphi\|_{2}
\end{aligned}
$$

where $\gamma$ is the constant of the embedding $L^{\frac{6}{5}}\left(E ; \mathbb{R}^{3}\right) \subset V$. Therefore, also the second map is a bounded linear functional in $V$. By the Riesz representation theorem ${ }^{3}$ there exists $\bar{B}(\mathbf{u}) \in V$, such that

[^5]$$
\int_{E}[\mathbf{u} \cdot(\mathbf{u} \cdot \nabla)+\mathbf{u} \cdot(\mathbf{b} \cdot \nabla)+\mathbf{b} \cdot(\mathbf{u} \cdot \nabla)] \boldsymbol{\varphi} d x=\langle\bar{B}(\mathbf{u}), \boldsymbol{\varphi}\rangle_{V}
$$

With these identifications the weak form (4.5) reads

$$
V \ni \boldsymbol{\varphi} \rightarrow \nu\langle\mathbf{u}, \boldsymbol{\varphi}\rangle_{V}=\langle\bar{B}(\mathbf{u})+\mathbf{G}, \boldsymbol{\varphi}\rangle_{V} .
$$

Equivalently, in functional form

$$
\begin{equation*}
\mathbf{u}=\overline{\mathcal{B}}(\mathbf{u}) \quad \text { in } \quad V^{*} \quad \text { where } \quad \overline{\mathcal{B}}(\mathbf{u})=\frac{1}{\nu}(\bar{B}(\mathbf{u})+\mathbf{G}) \tag{4.11}
\end{equation*}
$$

where, as before, $V^{*}$ denotes the dual of $V$ identified with $V$ itself up to an isometric isomorphism. Thus, existence of weak solution to (4.1) in the sense of (4.5) is equivalent to finding a fixed point of the map $V \ni \mathbf{u} \rightarrow \overline{\mathcal{B}}(\mathbf{u}) \in V^{*}$.
Lemma 4.1 The map $\overline{\mathcal{B}}(\cdot): V \rightarrow V^{*}$ is compact.
The proof is analogous to that of Lemma 3.1 with minor changes. Consider next the family of variants of (4.11)

$$
\begin{equation*}
\mathbf{u}=\lambda \mathcal{B}(\mathbf{u}) \quad \text { in } V \quad \text { for } \quad \lambda \in(0,1) \tag{4.12}
\end{equation*}
$$

If $\mathbf{u}_{\lambda}$ is a solution of (4.12), it is also a solution of (4.5) with $\nu$ replaced by $\nu / \lambda$. As such, the apriori estimate (4.10) remains in force with $\mathbf{u}$ replaced by $\mathbf{u}_{\lambda}$ and $\nu$ replaced by $\nu / \lambda$, i.e.,

$$
\left\|\mathbf{u}_{\lambda}\right\|_{V} \leq \frac{1}{\gamma_{o}}\left\|\nabla \mathbf{u}_{\lambda}\right\|_{2} \leq 2 \frac{\lambda}{\nu} \frac{\gamma}{\gamma_{o}}\left[\|\mathbf{f}\|_{\frac{6}{5}}+\nu\|\nabla \mathbf{b}\|_{2}+\|\mathbf{b}\|_{4}^{2}\right] .
$$

Therefore, all possible solutions of (4.12) are uniformly bounded in $\lambda$. Existence of solutions of (4.11), and hence of (4.1), now follows from the SchauderLeray Fixed Point Theorem 3.1.

## 5 Recovering the Pressure

Return to the steady-state Navier-Stokes system (2.1) in its weak form (2.3). Existence of solutions to such a system has been established in Section 3 irrespective of the pressure $p$ appearing in the formal pointwise form (2.1). Assume momentarily that

$$
\begin{equation*}
\mathbf{v} \in W^{2,2}\left(E ; \mathbb{R}^{3}\right) \quad \text { and } \quad \mathbf{f} \in L^{2}\left(E ; \mathbb{R}^{3}\right) \tag{5.1}
\end{equation*}
$$

Then (2.3) by back-integration by parts yields

$$
\int_{E}(\mathbf{N S}) \cdot \boldsymbol{\varphi} d x=0 \quad \text { where } \quad(\mathbf{N S})=-\nu \Delta \mathbf{v}+(\mathbf{v} \cdot \nabla) \mathbf{v}-\mathbf{f}
$$

for all $\varphi \in \mathcal{V}$. Since (NS) $\in L^{2}\left(E ; \mathbb{R}^{3}\right)$ this continues to hold for all $\varphi \in H$. Therefore, $(\mathbf{N S}) \in H^{\perp}$. Introduce the space of functions

$$
G=\left\{\begin{array}{c}
\text { collection of } \varphi \in L^{2}\left(E ; \mathbb{R}^{3}\right) \text { of the form } \\
\varphi=\nabla p \text { for some } p \in W^{1,2}(E)
\end{array}\right\}
$$

Proposition 5.1 (Helmholtz-Weyl Decomposition [54]) Let $E \subset \mathbb{R}^{3}$ be open, bounded and convex. Then $G=H^{\perp}$ or equivalently

$$
L^{2}\left(E ; \mathbb{R}^{3}\right)=H \oplus G
$$

Indeed, Proposition 5.1 is a special case of the Helmholtz-Weyl decomposition; its proof will be given in Section 5c of the Complements.

The system (2.1), as such, does not provide sufficient information to determine the pressure $p$. However, its weak formulation (2.3) permits one to assert that the principal part (NS) of the Navier-Stokes system has, at least under the regularity assumptions (5.1) on $\mathbf{v}$ and $\mathbf{f}$, and locally in $E$, the form of a gradient of some pressure $p \in W_{\mathrm{loc}}^{1,2}(E)$. This follows by applying Proposition 5.1 to open, convex subsets of $E$.

## 6 Steady State Flows in Unbounded Domains

Let $E$ be an unbounded, open set in $\mathbb{R}^{3}$ filled with a fluid of dynamic viscosity $\mu$. The problem is particularly interesting from the physical point of view if $E$ is an exterior domain, that is, the complement of a bounded set; such a situation can then be used to model the motion of a rigid body through a viscous fluid, or the flow past an obstacle (see also [15, Chapter 1, § 2] for more details).

The domain $E$ will be assumed to be open and simply connected, with boundary $\partial E$ of class $C^{1}$, and satisfying the segment property. The fluid velocity $\mathbf{v}$ is assumed to take the value a on $\partial E$, for a vector field a whose regularity will be specified as we proceed, and to approach a constant vector $\mathbf{a}_{\infty}$ as $|x| \rightarrow \infty$. The fluid is stirred in its interior by a forcing term $\mathbf{f}$ whose properties are to be defined. Consider formally the steady state flow in $E$,

$$
\begin{align*}
-\nu \Delta \mathbf{v}+(\mathbf{v} \cdot \nabla) \mathbf{v}+\nabla p & =\mathbf{f}, \\
\operatorname{div} \mathbf{v} & =0, \\
\left.\mathbf{v}\right|_{\partial E} & =\mathbf{a},  \tag{6.1}\\
\lim _{|x| \rightarrow \infty} \mathbf{v}(x) & =\mathbf{a}_{\infty} .
\end{align*}
$$

Notice that, in general, (4.2) is no longer a necessary condition on a for the solvability of (6.1), even if $\mathbf{a}_{\infty}=0$.

### 6.1 Assumptions on a and f

It is assumed that the boundary datum $\mathbf{a} \in W^{\frac{1}{2}, 2}\left(\partial E ; \mathbb{R}^{3}\right)$ can be extended into a solenoidal $\mathbf{b} \in W_{\text {loc }}^{1,2}\left(E ; \mathbb{R}^{3}\right)$, satisfying

$$
\begin{align*}
& \mathbf{b}=\mathbf{a} \quad \text { on } \partial E \text { as traces of functions in } W^{1,2}\left(E ; \mathbb{R}^{3}\right), \\
& \mathbf{b}-\mathbf{a}_{\infty} \in L^{2}\left(E ; \mathbb{R}^{3}\right) \\
& \left|\mathbf{b}(x)-\mathbf{a}_{\infty}\right| \leq \frac{M_{o}}{\sqrt{1+|x|^{2}}}, \quad \text { and }|\nabla \mathbf{b}| \leq \frac{M_{1}}{1+|x|^{2}} \quad \text { in } E, \tag{6.2}
\end{align*}
$$

for two given constants $M_{o}$ and $M_{1}$. For exterior domains and smooth a with zero flux on $\partial E$, such an extension can always be realized. Indeed we have the following.

Proposition 6.1 Let $E$ be an exterior domain, complement of a bounded, simply connected domain $E^{c}=\mathbb{R}^{3} \backslash \bar{E}$. Then any $\mathbf{a} \in C^{2}\left(\partial E ; \mathbb{R}^{3}\right)$ satisfying (4.2) admits a solenoidal extension $\mathbf{b} \in C^{2}\left(\mathbb{R}^{3} ; \mathbb{R}^{3}\right)$ satisfying (6.2).

Proof. For $\delta>0$, consider the set $E_{\delta}=[\operatorname{dist}(\cdot, \partial E)<\delta]$ and construct the vector field $\psi_{\mathbf{a}} \in C^{3}\left(\mathbb{R}^{3} ; \mathbb{R}^{3}\right)$ corresponding to a, compactly supported in $E_{\delta}$, such that the solenoidal extension $\mathbf{b}_{\mathbf{a}}$ of $\mathbf{a}$ is realized by $\mathbf{b}_{\mathbf{a}}=\operatorname{curl} \boldsymbol{\psi}$. Such a construction is guaranteed by Proposition 4.3c of the Complements.

Let $R>1$ be sufficiently large, such that $B_{R-1} \supset E^{c}$, and set

$$
\begin{equation*}
\mathbf{b}^{\prime}=\operatorname{curl}\left[\left(x_{3} a_{\infty, 2}, x_{1} a_{\infty, 3}, x_{2} a_{\infty, 1}\right) \zeta\right] \tag{6.3}
\end{equation*}
$$

where

$$
\zeta=\left\{\begin{array}{l}
1 \text { outside a ball of radius } R \\
0 \text { inside a ball of radius } R-1 \\
\text { smooth, } 0 \leq \zeta \leq 1 \text { otherwise }
\end{array}\right.
$$

Finally, let $\mathbf{b}(x)=\mathbf{b}_{\mathbf{a}}(x)+\mathbf{b}^{\prime}(x)$. One verifies that such a $\mathbf{b}$ is solenoidal, and satisfies the requirements (6.2).

For general vector fields with the regularity assumed in (6.2), again one relies on Proposition 4.3c of the Complements for the construction of $\mathbf{b}_{\mathbf{a}}$, whereas $\mathbf{b}^{\prime}$ is built as in (6.3).

By the previous construction, it is also apparent that $\operatorname{supp} \nabla \mathbf{b}$ is a compact set in $\mathbb{R}^{3}$.

The forcing term $\mathbf{f}$ is taken in $L_{\text {loc }}^{2}\left(E ; \mathbb{R}^{3}\right)$ and decreasing sufficiently fast as $|x| \rightarrow \infty$, in the sense

$$
\begin{equation*}
|x| \mathbf{f} \in L^{2}\left(E ; \mathbb{R}^{3}\right) \tag{6.4}
\end{equation*}
$$

### 6.2 Towards a Notion of Solution to (6.1)

Proceeding as in the case of bounded domains, solutions are sought of the form $\mathbf{v}=\mathbf{b}+\mathbf{u}$, for some $\mathbf{u} \in V$, where formally $\mathbf{u}$ satisfies (4.3)-(4.5), the latter holding for all $\varphi \in \mathcal{V}$. The membership $\mathbf{u} \in V$ provides weak forms of the last two conditions in (6.1), whereas (4.5) interprets weakly the NavierStokes system. The next step is in deriving apriori estimates for $\mathbf{u}$, by taking $\boldsymbol{\varphi}=\mathbf{u}$ in (4.5).

For bounded domains $E$, the inner product $(\cdot, \cdot)_{V}$ introduced in (3.1) is equivalent to the inner product $\langle\nabla \cdot, \nabla \cdot\rangle_{H}$. This follows from the embedding inequalities (2.5)-(2.6). If $E$ is unbounded this is, in general, no longer the case and the topology generated by $(\cdot, \cdot)_{V}$ cannot be related to the norm $\|\nabla \cdot\|_{2}$. Nevertheless the first of (2.5) has a weaker counterpart in $V$.

Proposition 6.2 Let $\mathbf{u} \in V$ and $0 \in \mathbb{R}^{3} \backslash \operatorname{supp} \mathbf{u}$. Then

$$
\begin{equation*}
\int_{E} \frac{|\mathbf{u}|^{2}}{|x|^{2}} d x \leq 4 \int_{E}|\nabla \mathbf{u}|^{2} d x \tag{6.5}
\end{equation*}
$$

Proof. Since u has vanishing trace on $\partial E$, by extending it with zero outside $E$, regard $\mathbf{u}$ as an element of $V$ in $\mathbb{R}^{3}$. Assume momentarily that $\mathbf{u} \in \mathcal{V}$, and for $\varepsilon \in(0,1)$ compute and estimate

$$
\begin{aligned}
\int_{\varepsilon<|x|<\varepsilon^{-1}} \frac{|\mathbf{u}|^{2}}{|x|^{2}} d x & =\int_{\varepsilon<|x|<\varepsilon^{-1}}(\Delta \ln |x|)|\mathbf{u}|^{2} d x \\
& =\int_{|x|=\varepsilon^{-1}} \nabla \ln |x| \cdot \frac{x}{|x|}|\mathbf{u}|^{2} d \sigma-\int_{|x|=\varepsilon} \nabla \ln |x| \cdot \frac{x}{|x|}|\mathbf{u}|^{2} d \sigma \\
& -2 \int_{\varepsilon<|x|<\varepsilon^{-1}} \frac{\mathbf{u}}{|x|} \cdot\left(x \cdot \frac{\nabla \mathbf{u}}{|x|}\right) d x
\end{aligned}
$$

Letting $\varepsilon \rightarrow 0$ gives

$$
\int_{E} \frac{|\mathbf{u}|^{2}}{|x|^{2}} d x \leq 2 \int_{E} \frac{|\mathbf{u}|}{|x|}|\nabla \mathbf{u}| d x \leq\left(4 \int_{E} \frac{|\mathbf{u}|^{2}}{|x|^{2}} d x\right)^{\frac{1}{2}}\left(\int_{E}|\nabla \mathbf{u}|^{2} d x\right)^{\frac{1}{2}}
$$

which yields (6.5). The proof is then concluded by density.
Notice that in the proof it is not used that $\mathbf{u}$ is solenoidal.

## 7 Existence of Solutions to (6.1)

### 7.1 Approximating Solutions and Apriori Estimates

For $n>\operatorname{diam}\left(E^{c}\right)$ let $B_{n}$ be the ball of radius $n$ about the origin of $\mathbb{R}^{3}$, set $E_{n}=E \cap B_{n}$, and let $\mathcal{V}_{n}$ and $V_{n}$ be the spaces introduced in (2.2) for $E_{n}$.

In each $E_{n}$ we consider the problem

$$
\begin{aligned}
-\nu \Delta \mathbf{v}_{n}+\left(\mathbf{v}_{n} \cdot \nabla\right) \mathbf{v}_{n}+\nabla p & =\mathbf{f} \\
\operatorname{div} \mathbf{v}_{n} & =0 \\
\left.\mathbf{v}_{n}\right|_{\partial E} & =\mathbf{a} \\
\left.\mathbf{v}_{n}\right|_{\partial B_{n}} & =\left.\mathbf{b}\right|_{\partial B_{n}} .
\end{aligned}
$$

Let $\mathbf{v}_{n}=\mathbf{b}+\mathbf{u}_{n}$, where $\mathbf{u}_{n} \in V_{n}$ is a weak solution in $E_{n}$ of

$$
\begin{align*}
-\nu \Delta \mathbf{u}_{n}+\left(\mathbf{u}_{n} \cdot \nabla\right) \mathbf{u}_{n}+(\mathbf{b} \cdot \nabla) \mathbf{u}_{n}+\left(\mathbf{u}_{n} \cdot \nabla\right) \mathbf{b}+\nabla p & =\mathbf{g}, \\
\operatorname{div} \mathbf{u}_{n} & =0 \\
\left.\mathbf{u}_{n}\right|_{\partial E} & =\mathbf{0}  \tag{7.1}\\
\left.\mathbf{u}_{n}\right|_{\partial B_{n}} & =\mathbf{0}
\end{align*}
$$

where $\mathbf{g}=\mathbf{f}+\nu \Delta \mathbf{b}-(\mathbf{b} \cdot \nabla) \mathbf{b}$. Since $E_{n}$ is bounded, such a $\mathbf{u}_{n}$ exists, by the construction in Section 4.

Proposition 7.1 Let $\mathbf{u}_{n} \in V_{n}$ be a solution of (7.1) in $E_{n}$, with $\mathbf{b}$ and $\mathbf{f}$ satisfying (6.2) and (6.4) respectively. Then either of these apriori estimates holds, uniformly in $n$

$$
\begin{align*}
& \left(\nu-2 M_{o}\right)\left\|\nabla \mathbf{u}_{n}\right\|_{2} \leq \gamma_{\mathbf{g}} \stackrel{\text { def }}{=}\||x| \mathbf{f}\|_{2}+\pi\left(C \pi M_{1}^{2}+\nu M_{1}\right), ~  \tag{7.2}\\
& \left(\nu-4 M_{1}\right)\left\|\nabla \mathbf{u}_{n}\right\|_{2}
\end{align*}
$$

where $C>0$ is a constant that depends on $\delta$ and $R$, introduced in the proof of Proposition 6.1.

Proof. Since $\mathbf{u}_{n} \in V_{n}$, it can be regarded as $\mathbf{u}_{n} \in V$, by extending it to be zero outside $E_{n}$. Likewise, the test function $\varphi \in V_{n}$ is regarded as in $V$. Insert $\boldsymbol{\varphi}=\mathbf{u}$ in the weak formulation of (7.1), and transform and estimate the various terms by using the assumptions (6.4) on $\mathbf{f}$ and (6.2) on $\mathbf{b}$. In this process we use the elementary calculation

$$
\int_{\mathbb{R}^{3}} \frac{d x}{\left(1+|x|^{2}\right)^{2}}=\pi^{2}
$$

We have

$$
\begin{aligned}
\nu \int_{E_{n}} \nabla \mathbf{u}_{n}: \nabla \mathbf{u}_{n} d x \leq & \left|\int_{E_{n}} \mathbf{u}_{n} \cdot(\mathbf{b} \cdot \nabla) \mathbf{u}_{n} d x\right| \\
& +\left|\int_{E_{n}} \mathbf{u}_{n} \cdot\left(\mathbf{u}_{n} \cdot \nabla\right) \mathbf{b} d x\right| \\
& +\left|\int_{E_{n}} \mathbf{f} \cdot \mathbf{u}_{n} d x\right| \\
& +\left|\nu \int_{E_{n}} \Delta \mathbf{b} \cdot \mathbf{u}_{n} d x\right| \\
& +\left|\int_{E_{n}}(\mathbf{b} \cdot \nabla) \mathbf{b} \cdot \mathbf{u}_{n} d x\right|
\end{aligned}
$$

The various terms above are transformed and estimated as follows:

Combining these calculations proves (7.2).
The proposition provides an apriori estimate for $\left\|\nabla \mathbf{u}_{n}\right\|_{2}$, independent of $n$ if either $M_{o}$ or $M_{1}$ are sufficiently small. In what follows assume

$$
\begin{equation*}
\max \left\{2 M_{o} ; 4 M_{1}\right\}<\nu \quad \text { and set } \quad \nu-\max \left\{2 M_{o} ; 4 M_{1}\right\}=\alpha>0 . \tag{7.3}
\end{equation*}
$$

For unbounded $E$, introduce the space

$$
\mathcal{H}=\left\{\text { completion of } \mathcal{V} \text { in the norm }\|\cdot\|_{\mathcal{H}}=\|\nabla \cdot\|_{2}\right\} .
$$

This is a separable Hilbert space by the inner product $\langle\cdot, \cdot\rangle_{\mathcal{H}}=\langle\nabla \cdot, \nabla \cdot\rangle_{H}$. By construction $V \subset \mathcal{H}$, as elements in $\mathcal{H}$ are not required to be in $L^{2}\left(E ; \mathbb{R}^{3}\right)$.

### 7.2 The Limiting Process

If (7.3) holds, then $\left\{\mathbf{u}_{n}\right\}$ is a sequence bounded in $\mathcal{H}$, and hence weakly precompact in the same space. Every element $\mathbf{u}$ in the weak closure of $\left\{\mathbf{u}_{n}\right\}$ is a weak solution of (4.5) in the following sense. First $\mathbf{u} \in \mathcal{H}^{*}$ and hence $\mathbf{u} \in \mathcal{H}$ by the Riesz identification map. Next, having fixed $\varphi \in \mathcal{V}$, let $F$ be its support and consider the sequence $\left\{\left.\mathbf{u}_{n}\right|_{F}\right\}$ of restrictions of $\mathbf{u}_{n}$ to $F$. Since $F$ is bounded, by the embedding inequalities (2.5)-(2.6), the norm $\|\nabla \cdot\|_{2 ; F}$ is equivalent to the norm of $W^{1,2}\left(F ; \mathbb{R}^{3}\right)$. Therefore, there exists a constant $C$, depending on $F$, such that

$$
\left\|\left.\mathbf{u}_{n}\right|_{F}\right\|_{W^{1,2}\left(F ; \mathbb{R}^{3}\right)} \leq C \quad \text { uniformly in } n
$$

By the Rellich-Kondrachov compact embedding theorem, the embedding $W^{1,2}\left(F ; \mathbb{R}^{3}\right) \hookrightarrow L^{p}\left(F ; \mathbb{R}^{3}\right)$ is compact for all $1 \leq p<6$. Therefore, a subsequence $\left\{\left.\mathbf{u}_{n^{\prime}}\right|_{F}\right\} \subset\left\{\left.\mathbf{u}_{n}\right|_{F}\right\}$ can be selected so that

$$
\begin{array}{ll}
\left\{\mathbf{u}_{n^{\prime}}\right\} \rightarrow \mathbf{u} & \text { weakly in } W^{1,2}\left(F ; \mathbb{R}^{3}\right), \text { and } \\
\left\{\mathbf{u}_{n^{\prime}}\right\} \rightarrow \mathbf{u} & \text { strongly in } L^{r}\left(F ; \mathbb{R}^{3}\right) \tag{7.4}
\end{array}
$$

Theorem 7.1. Let (6.2), (6.3) and (7.3) hold. Then (4.5) admits a solution $\mathbf{u} \in \mathcal{H}$ satisfying

$$
\begin{equation*}
\nu \int_{E} \nabla \mathbf{u}: \nabla \varphi d x=\int_{E}\{[\mathbf{u} \cdot(\mathbf{u} \cdot \nabla)+\mathbf{u} \cdot(\mathbf{b} \cdot \nabla)+\mathbf{b} \cdot(\mathbf{u} \cdot \nabla)] \boldsymbol{\varphi}+\mathbf{g} \cdot \boldsymbol{\varphi}\} d x \tag{7.5}
\end{equation*}
$$

for all $\varphi \in \mathcal{V}$.
Proof. Let $\mathbf{u} \in \mathcal{H}$ be in the closure of $\left\{\mathbf{u}_{n}\right\}$ in the weak topology of $\mathcal{H}$. Write down (4.5) for $\mathbf{u}_{n}$ over $E_{n}$ for $\varphi_{n} \in \mathcal{V}_{n}$. Having fixed $\varphi \in \mathcal{V}$, let $F$ be its support and let $n_{F}$ be so large that $F \subset \subset B_{n}$ for all $n \geq n_{F}$. Then for such $\varphi$ fixed, (4.5) will hold for all $n \geq n_{F}$. Letting $n \rightarrow \infty$ along proper subsequences depending of $\varphi$ satisfying (7.4) establishes (7.5)
Remark 7.1 Notice that the indicated limiting process can be carried out for a fixed $\varphi$ of compact support and not for $\varphi \in V$. Thus (7.5) holds only for $\varphi \in \mathcal{V}$ and, in general, not for $\varphi \in V$. Once $\mathbf{u}$ in the weak closure of $\left\{\mathbf{u}_{n}\right\}$ has been identified, the choice of subsequences for which (7.4) holds depends on the selected testing function $\varphi$. However, the limiting identity (7.5) continues to hold for all $\varphi \in \mathcal{V}$. Also, for unbounded $E$, solutions are found in $\mathcal{H}$ and in general not in $V$.

## 8 Time-Dependent Navier-Stokes Equations in Bounded Domains

Continue to denote by $E \subset \mathbb{R}^{3}$ an open, bounded set with boundary $\partial E$ of class $C^{1}$ and satisfying the segment property. For $0<T<\infty$, let $E_{T}=$ $E \times(0, T)$, and introduce the spaces

$$
\begin{aligned}
L^{2}(0, T ; V) & =\left\{\mathbf{v}(\cdot, t) \in V \text { for a.e. } t \in(0, T) \text { with finite norm }\|\nabla \mathbf{v}\|_{2 ; E_{T}}\right\} \\
W & =\left\{\begin{array}{c}
\mathbf{v}(\cdot, t) \in V \text { for a.e. } t \in(0, T) \text { with finite norm } \\
\|\mathbf{v}\|_{W}^{2}=\operatorname{esssup}_{(0, T)}\|\mathbf{v}(\cdot, t)\|_{2 ; E}^{2}+\|\nabla \mathbf{v}\|_{2 ; E_{T}}^{2}
\end{array}\right\} \\
C^{\infty}(0, T ; \mathcal{V}) & =\left\{\varphi \in C^{\infty}\left(E_{T} ; \mathbb{R}^{3}\right) \text { with } \varphi(\cdot, t) \in \mathcal{V} \text { for all } t \in(0, T)\right\}
\end{aligned}
$$

For these spaces the operations of $\nabla$ and div are meant weakly and with respect to the space variables only. Functions $\varphi \in C^{\infty}(0, T ; \mathcal{V})$ are divergence free and of compact support in $E$, in the space variables, but are permitted not to vanish for $t=0$ or for $t=T$.
Lemma 8.1 Let $\mathbf{v} \in W$. Then $\mathbf{v} \in L^{\frac{10}{3}}\left(E_{T} ; \mathbb{R}^{3}\right)$ and

$$
\|\mathbf{v}\|_{\frac{10}{3} ; E_{T}} \leq \gamma^{\frac{3}{5}}\|\mathbf{v}\|_{W}
$$

where $\gamma$ is the constant of the embedding of $V$ into $L^{6}\left(E ; \mathbb{R}^{3}\right)$.
Proof.

$$
\begin{aligned}
\int_{0}^{T} \int_{E}|\mathbf{v}|^{\frac{10}{3}} d x d t & =\int_{0}^{T} \int_{E}|\mathbf{v}|^{\frac{4}{3}}|\mathbf{v}|^{2} d x d t \\
& \leq \int_{0}^{T}\left(\int_{E}|\mathbf{v}|^{2} d x\right)^{\frac{2}{3}}\left(\int_{E}|\mathbf{v}|^{6} d x\right)^{\frac{1}{3}} d t \\
& \leq\left(\underset{(0, T)}{\operatorname{ess} \sup }\|\mathbf{v}(\cdot, t)\|_{2 ; E}\right)^{\frac{4}{3}} \int_{0}^{T}\|\mathbf{v}(\cdot, t)\|_{6 ; E}^{2} d t \\
& \leq \gamma^{2}\|\mathbf{v}\|_{W}^{\frac{10}{3}}
\end{aligned}
$$

The last inequality follows from the embedding (2.4).
Consider a viscous fluid of Reynolds number $\nu^{-1}$ filling a rigid, still container $E$ and stirred by a forcing term $\mathbf{f}$. Its time evolution over $(0, T)$ is modeled, formally, by the system

$$
\begin{align*}
\mathbf{v}_{t}-\nu \Delta \mathbf{v}+(\mathbf{v} \cdot \nabla) \mathbf{v}+\nabla p & =\mathbf{f} \quad \text { in } E_{T} ; \\
\operatorname{div} \mathbf{v} & =0 ; \\
\left.\mathbf{v}(\cdot, t)\right|_{\partial E} & =0 ;  \tag{8.1}\\
\mathbf{v}(\cdot, 0) & =\mathbf{v}_{o} \quad \text { in } E .
\end{align*}
$$

The homogeneous boundary condition for the velocity $\mathbf{v}$, also called no-slip condition, says that at the boundary, the fluid will have zero velocity with respect to the same boundary.

Multiply the first of these, formally, by $\varphi \in C^{\infty}(0, T ; \mathcal{V})$ and integrate by parts over $E_{t}$ for $t \in(0, T]$. Using that $\operatorname{div} \mathbf{v}=0$ gives

$$
\begin{align*}
\int_{E} \mathbf{v}(t) \cdot \boldsymbol{\varphi}(t) d x & -\int_{0}^{t} \int_{E} \mathbf{v} \cdot \boldsymbol{\varphi}_{\tau} d x d \tau \\
& +\int_{0}^{t} \int_{E}(\nu \nabla \mathbf{v}: \nabla \boldsymbol{\varphi}+(\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \boldsymbol{\varphi}) d x d \tau  \tag{8.2}\\
& =\int_{E} \mathbf{v}_{o} \cdot \boldsymbol{\varphi}(0) d x+\int_{0}^{t} \int_{E} \mathbf{f} \cdot \boldsymbol{\varphi} d x d \tau
\end{align*}
$$

Assuming momentarily that $\mathbf{v} \in C^{\infty}(0, T ; \mathcal{V})$ take $\boldsymbol{\varphi}=\mathbf{v}$ and observe that the non-linear term gives, formally, zero contribution. Using also Lemma 8.1 yields the formal energy inequality

$$
\begin{align*}
\underset{(0, T)}{\operatorname{ess} \sup } & \|\mathbf{v}(t)\|_{2 ; E}^{2}+2 \nu\|\nabla \mathbf{v}\|_{2 ; E_{T}}^{2} \\
& \leq\left\|\mathbf{v}_{o}\right\|_{2 ; E}^{2}+2 \int_{0}^{T} \int_{E} \mathbf{f} \cdot \mathbf{v} d x d t  \tag{8.3}\\
& \leq\left\|\mathbf{v}_{o}\right\|_{2 ; E}^{2}+2\|\mathbf{f}\|_{2 ; E_{T}}\|\mathbf{v}\|_{2 ; E_{T}} \\
& \leq\left\|\mathbf{v}_{o}\right\|_{2 ; E}^{2}+2 \sqrt{T}\|\mathbf{f}\|_{2 ; E_{T}} \underset{(0, T)}{\operatorname{ess} \sup }\|\mathbf{v}(\cdot, t)\|_{2 ; E}
\end{align*}
$$

In what follows, the set of parameters $\left\{\nu, T,|E|,\left\|\mathbf{v}_{o}\right\|_{2 ; E},\|\mathbf{f}\|_{2 ; E_{T}}\right\}$ are the given data and we will denote by $\gamma$ a generic positive constant that can be determined quantitatively, apriori only in terms of these. With this notation, by a standard application of the Cauchy-Schwarz inequality, (8.3) implies

$$
\begin{equation*}
\|\mathbf{v}\|_{W} \leq \gamma\left(\left\|\mathbf{v}_{o}\right\|_{2 ; E}+\|\mathbf{f}\|_{2 ; E_{T}}\right) \tag{8.4}
\end{equation*}
$$

These formal remarks suggest we define a weak solution to (8.1) as an element of $W$ satisfying (8.2) for all $\boldsymbol{\varphi} \in C^{\infty}(0, T ; \mathcal{V})$, and the energy estimate (8.4). The membership $\mathbf{v}(\cdot, t) \in V$ for a.e. $t \in(0, T)$ gives meaning, in the sense of traces, to the homogeneous boundary data on $\partial E$. The same membership insures that $\operatorname{div} \mathbf{v}=0$ weakly in $E_{T}$. As for the initial data, observe that, for solutions in this class, all integrals in (8.2) are well defined. As a consequence, by Vitali's absolute continuity of the integral, all integrals extended over $E_{t}$ tend to zero as $t \rightarrow 0$. Therefore,

$$
\lim _{t \rightarrow 0} \int_{E} \mathbf{v}(t) \cdot \boldsymbol{\varphi}(t) d x=\int_{E} \mathbf{v}_{o} \cdot \boldsymbol{\varphi}(0) d x \quad \text { for all } \boldsymbol{\varphi} \in C^{\infty}(0, T ; \mathcal{V})
$$

Thus the initial datum $\mathbf{v}_{o}$ is taken in the sense of such a weak continuity of $\mathbf{v}(\cdot, t)$ in $L^{2}\left(E ; \mathbb{R}^{3}\right)$. The same continuity also implies that $\operatorname{div} \mathbf{v}_{o}=0$ weakly in $E$. The latter emerges then as a compatibility condition to be imposed on the initial datum $\mathbf{v}_{o}$ for a solution to exist.
Theorem 8.1 (Hopf [20]). Let $\mathbf{f} \in L^{2}\left(E_{T} ; \mathbb{R}^{3}\right)$ and let $\mathbf{v}_{o} \in L^{2}\left(E ; \mathbb{R}^{3}\right)$ be weakly divergence free in $E$. Then there exists a weak solution to (8.1).
Remark 8.1 In the following we refer to such a solution as Hopf's solution.

## 9 The Galerkin Approximations

Let $\mathbf{e}=\left(\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}, \ldots\right)$ be a complete system for $V$. Since $\mathcal{V}$ is dense in $V$, by sequential selection and Zorn's lemma, the elements $\mathbf{e}_{j}$ can be chosen in $\mathcal{V}$. Also by sequential orthonormalization, while not necessarily orthonormal with respect to the inner product $\langle\cdot, \cdot\rangle_{V}$ defined in (3.1), they can be chosen to be orthonormal in $L^{2}\left(E ; \mathbb{R}^{3}\right)$, i.e., $\left\langle\mathbf{e}_{i}, \mathbf{e}_{j}\right\rangle_{H}=\delta_{i j}$. Write a possible solution in the form

$$
\begin{equation*}
\mathbf{v}=\mathbf{v}_{n}+\mathbf{v}_{r, n} \quad \text { where } \quad \mathbf{v}_{n}=\sum_{j=1}^{n} c_{j}(t) \mathbf{e}_{j} \quad \text { and } \quad \mathbf{v}_{r, n}=\sum_{j>n} c_{j}(t) \mathbf{e}_{j} \tag{9.1}
\end{equation*}
$$

for scalar functions $(0, T) \ni t \rightarrow c_{j}(t)$. The remainder $\mathbf{v}_{r, n}$ of the series satisfies

$$
\left\|\mathbf{v}_{r, n}\right\|_{V}^{2}=\left\|\mathbf{v}_{r, n}\right\|_{2}^{2}+\left\|\nabla \mathbf{v}_{r, n}\right\|_{2}^{2} \rightarrow 0 \quad \text { as } n \rightarrow \infty
$$

Since $\left(\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}, \ldots\right)$ is complete in $V$, it is also complete in $L^{2}\left(E ; \mathbb{R}^{3}\right)$. Therefore, by the indicated orthonormalization in $L^{2}\left(E ; \mathbb{R}^{3}\right)$ and Parseval's identity

$$
\|\mathbf{v}\|_{2 ; E}^{2}=\sum_{j \geq 1} c_{j}^{2}
$$

Write $\mathbf{v}$ in (8.2) in the form (9.1) and observe that the terms involving $\mathbf{v}_{r, n}$ tend to zero as $n \rightarrow \infty$. This suggests defining an approximate solution to (8.1) as a function $\mathbf{v}_{n} \in C^{\infty}(0, T ; \mathcal{V})$, with $\mathbf{v}_{n}=\sum_{i=1}^{n} c_{n, i} \mathbf{e}_{i}$, satisfying (8.2) for $\varphi=\mathbf{e}_{i}$, for all $i=1, \ldots, n$, i.e.,

$$
\begin{align*}
\int_{0}^{T}\left[c_{n, i}^{\prime}\right. & +\sum_{j=1}^{n} c_{n, j}\left\{\int_{E} \nu \nabla \mathbf{e}_{j}: \nabla \mathbf{e}_{i} d x\right\}_{i j}^{\mathrm{sym}} \\
& \left.+\sum_{j=1}^{n} c_{n, j}\left\{\int_{E} \mathbf{e}_{i} \cdot\left(\mathbf{v}_{n} \cdot \nabla\right) \mathbf{e}_{j} d x\right\}_{i j}^{\text {skew }} d \tau-\int_{E} \mathbf{f} \cdot \mathbf{e}_{i} d x\right] d \tau=0 \tag{9.2}
\end{align*}
$$

For fixed $n \in \mathbb{N}$ the terms $A_{i j}^{\text {sym }}=\{\cdots\}_{i j}^{\text {sym }}$ define the entries of a $n \times n$ time independent symmetric matrix $\mathbf{A}_{n}^{\text {sym }}$, whereas the terms $A_{i j}^{\text {skew }}=\{\cdots\}_{i j}^{\text {skew }}$ define the entries of a $n \times n$ skew symmetric matrix $\mathbf{A}_{n}^{\text {skew }}$ linearly dependent on the time dependent vector $\mathbf{c}_{n}=\left(c_{n, 1}, \ldots, c_{n, n}\right)$. The last term defines a vector $\mathbf{f}_{n}=\left(f_{1}, \ldots, f_{n}\right)$ dependent on $t$. Set also

$$
c_{o, i}=\int_{E} \mathbf{v}_{o} \cdot \mathbf{e}_{i} d x, \quad \mathbf{c}_{o}=\left(c_{o, 1}, \ldots, c_{o, n}\right), \quad \mathbf{c}_{n}(0)=\mathbf{c}_{o}
$$

Requiring that the integrand over $(0, T)$ in (9.2) vanishes identically, gives the differential system in $\mathbf{c}_{n}$

$$
\begin{equation*}
c_{n, i}^{\prime}+\sum_{j=1}^{n}\left(A_{i j}^{\text {sym }}+A_{i j}^{\text {skew }}\right) c_{n, j}=f_{i} \quad \text { with } \quad c_{n, i}(0)=c_{o, i} \tag{9.3}
\end{equation*}
$$

Unique solvability of this system hinges upon some apriori estimates which we derive next.

Proposition 9.1 Let $\mathbf{c}_{n}=\left(c_{n, 1}, \ldots, c_{n, n}\right)$ be a solution to (9.3) and set $\mathbf{v}_{n}=$ $\sum_{i=1}^{n} c_{n, i} \mathbf{e}_{i}$. There is a constant $\gamma$ depending only on the data and independent of $n$ and $i$, such that

$$
\begin{align*}
\left\|\mathbf{v}_{n}\right\|_{W} & \leq \gamma ; \\
\operatorname{ess} \sup & c_{n, i}(t) \mid \tag{9.4}
\end{align*} \leq \gamma ;(0, T)<c_{n, i}\left(t_{2}\right)-c_{n, i}\left(t_{1}\right) \mid \leq \gamma\left(1+\left\|\nabla \mathbf{e}_{i}\right\|_{\infty ; E}\right) \sqrt{t_{2}-t_{1}}
$$

for all $\left(t_{1}, t_{2}\right) \subset(0, T)$.
Proof. Multiply (9.3) by $c_{n, i}$, add over $i=1, \ldots, n$, and observe that $\mathbf{c}_{n}^{t} \mathbf{A}_{n}^{\text {skew }} \mathbf{c}_{n}=0$, where $\mathbf{c}_{n}^{t}$ denotes the transpose of the vector $\mathbf{c}_{n}$. This gives

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t} \sum_{i=1}^{n} c_{n, i}^{2}+\nu \int_{E} \nabla \sum_{j=1}^{n} c_{n, j} \mathbf{e}_{j}: \nabla \sum_{i=1}^{n} c_{n, i} \mathbf{e}_{i} d x \\
& \quad=\sum_{i=1}^{n} f_{i} c_{n, i} \leq\left(\sum_{i=1}^{n} f_{i}^{2}\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n} c_{n, i}^{2}\right)^{\frac{1}{2}}=\left\|\mathbf{f}_{n}(t)\right\|_{2 ; E}\left\|\mathbf{v}_{n}(t)\right\|_{2 ; E}
\end{aligned}
$$

Equivalently

$$
\frac{1}{2} \frac{d}{d t}\left\|\mathbf{v}_{n}(t)\right\|_{2 ; E}^{2}+\nu\left\|\nabla \mathbf{v}_{n}(t)\right\|_{2 ; E}^{2} \leq\left\|\mathbf{f}_{n}(t)\right\|_{2 ; E}\left\|\mathbf{v}_{n}(t)\right\|_{2 ; E}
$$

To prove the first of (9.4), integrate this over $(0, t) \subset(0, T)$ to get

$$
\begin{aligned}
\underset{(0, T)}{\operatorname{ess} \sup }\left\|\mathbf{v}_{n}(t)\right\|_{2 ; E}^{2} & +2 \nu\left\|\nabla \mathbf{v}_{n}\right\|_{2 ; E_{T}}^{2} \\
& \leq\left\|\mathbf{v}_{o}\right\|_{2 ; E}^{2}+2 \sqrt{T}\left\|\mathbf{f}_{n}\right\|_{2 ; E_{T}} \underset{(0, T)}{\operatorname{ess} \sup }\left\|\mathbf{v}_{n}(t)\right\|_{2 ; E} .
\end{aligned}
$$

The proof is concluded by a standard application of Cauchy-Schwarz inequality in the last term. The second of (9.4) follows from this and Parseval's identity. To prove the last of (9.4), return to (9.3) and, for fixed $i \in\{1, \ldots, n\}$, estimate

$$
\begin{aligned}
& \left|c_{n, i}^{\prime}\right| \leq \nu\left|\int_{E} \nabla \mathbf{v}_{n}: \nabla \mathbf{e}_{i} d x\right|+\left|\int_{E}\left(\mathbf{v}_{n} \cdot \nabla\right) \mathbf{e}_{i} \cdot \mathbf{v}_{n} d x\right|+\int_{E}|\mathbf{f}| d x \\
& \quad \leq \nu \int_{E}\left|\nabla \mathbf{v}_{n} \| \nabla \mathbf{e}_{i}\right| d x+\int_{E}\left|\mathbf{v}_{n}\right|^{2}\left|\nabla \mathbf{e}_{i}\right| d x+\int_{E}|\mathbf{f}| d x \\
& \quad \leq \nu\left\|\nabla \mathbf{e}_{i}\right\|_{\infty ; E} \int_{E}\left|\nabla \mathbf{v}_{n}\right| d x+\left\|\nabla \mathbf{e}_{i}\right\|_{\infty ; E} \int_{E}\left|\mathbf{v}_{n}\right|^{2} d x+\int_{E}|\mathbf{f}| d x \\
& \left.\quad \leq \nu\left\|\nabla \mathbf{e}_{i}\right\|_{\infty ; E}\left\|\nabla \mathbf{v}_{n}(t)\right\|_{2 ; E}|E|^{\frac{1}{2}}+\left\|\nabla \mathbf{e}_{i}\right\|_{\infty ; E} \underset{(0, T)}{\operatorname{ess} \sup }\left\|\mathbf{v}_{n}(t)\right\|_{2 ; E}^{2}\right) \\
& \quad+|E|^{\frac{1}{2}}\|\mathbf{f}(t)\|_{2 ; E} \\
& \quad \leq\left\|\nabla \mathbf{e}_{i}\right\|_{\infty ; E}\left(\nu|E|^{\frac{1}{2}}\left\|\nabla \mathbf{v}_{n}(t)\right\|_{2 ; E}+\underset{(0, T)}{\left.\underset{\operatorname{ess} \sup }{ }\left\|\mathbf{v}_{n}(t)\right\|_{2 ; E}^{2}\right)+|E|^{\frac{1}{2}}\|\mathbf{f}(t)\|_{2 ; E} .}\right.
\end{aligned}
$$

Integrating over $\left(t_{1}, t_{2}\right) \subset(0, T)$ and using the first of (9.4) gives

$$
\begin{aligned}
\left|c_{n, i}\left(t_{2}\right)-c_{n, i}\left(t_{1}\right)\right| \leq & \left|\int_{t_{1}}^{t_{2}} c_{n, i}^{\prime}(t) d t\right| \leq \int_{t_{1}}^{t_{2}}\left|c_{n, i}^{\prime}(t)\right| d t \\
\leq & \nu\left\|\nabla \mathbf{e}_{i}\right\|_{\infty ; E}|E|^{\frac{1}{2}} \int_{t_{1}}^{t_{2}}\left(\int_{E}\left|\nabla \mathbf{v}_{n}\right|^{2} d x\right)^{\frac{1}{2}} d t \\
& +\left\|\nabla \mathbf{e}_{i}\right\|_{\infty ; E} \underset{(0, T)}{\operatorname{ess} \sup }\left\|\mathbf{v}_{n}(t)\right\|_{2 ; E}^{2}\left(t_{2}-t_{1}\right) \\
& +|E|^{\frac{1}{2}} \int_{t_{1}}^{t_{2}}\left(\int_{E}|\mathbf{f}|^{2} d x\right)^{\frac{1}{2}} d t \\
& \leq \gamma\left(\nu, T,|E|,\left\|\mathbf{v}_{o}\right\|_{2},\|\mathbf{f}\|_{2 ; E_{T}}\right)\left(1+\left\|\nabla \mathbf{e}_{i}\right\|_{\infty}\right) \sqrt{t_{2}-t_{1}}
\end{aligned}
$$

Existence and uniqueness of solutions to (9.3) can be established in the small, for example by a contraction fixed point argument. Then the solution can be continued in the whole $(0, T)$, so long as it remains bounded. Such a bound, independent of $t, n$ and $i$, is insured by the second of (9.4).

## 10 Selecting Subsequences Strongly Convergent in $L^{2}\left(E_{T} ; \mathbb{R}^{3}\right)$

It follows from Proposition 9.1 that for fixed $j \in \mathbb{N}$ the sequences $\left\{c_{n, j}\right\}_{n=1}^{\infty}$ are equibounded and equicontinuous, so that by the Ascoli-Arzelà theorem a subsequence $\left\{c_{n_{j}, j}\right\} \subset\left\{c_{n, j}\right\}_{n=1}^{\infty}$ can be selected converging to some $c_{j}$ uniformly in $(0, T)$. By the Cantor diagonalization procedure a further subsequence can be selected and relabelled with $n$, such that $\left\{c_{n, j}\right\} \rightarrow c_{j}$ uniformly in $[0, T]$. However, it should be noted that, because of the last of (9.4), the rate of convergence depends on the index $j$. Set formally

$$
\mathbf{v}=\sum_{j=1}^{\infty} c_{j} \mathbf{e}_{j} .
$$

Proposition 10.1 For the same constant $\gamma$ as in the first of (9.4) there holds

$$
\underset{(0, T)}{\operatorname{ess} \sup }\|\mathbf{v}(\cdot, t)\|_{2 ; E}+\|\nabla \mathbf{v}\|_{2 ; E_{T}} \leq \gamma
$$

Moreover $\left\{\mathbf{v}_{n}(\cdot, t)\right\} \rightarrow \mathbf{v}(\cdot, t)$ weakly in $L^{2}\left(E ; \mathbb{R}^{3}\right)$, uniformly in $t \in(0, T)$.
Proof. For a fixed positive integer $k$ and all $n$

$$
\begin{aligned}
\sum_{j=1}^{k} c_{j}^{2}(t) & \leq\left|\sum_{j=1}^{k} c_{j}^{2}(t)-\sum_{j=1}^{k} c_{n, j}^{2}(t)\right|+\sum_{j=1}^{k} c_{n, j}^{2}(t) \\
& \leq \sum_{j=1}^{k}\left|c_{j}^{2}(t)-c_{n, j}^{2}(t)\right|+\left\|\mathbf{v}_{n}(\cdot, t)\right\|_{2 ; E}^{2}
\end{aligned}
$$

By the first of (9.4) the last term is bounded by a constant $\gamma$ depending only upon the data and independent of $t$ and $n$. Letting $n \rightarrow \infty$ the first term in the right-hand side tends to zero by the uniform convergence of $\left\{c_{n, j}\right\} \rightarrow c_{j}$ for $j=1, \ldots, k$. Thus $\sum_{j=1}^{k} c_{j}^{2}(t) \leq \gamma$. Since $k$ is arbitrary, the series $\sum_{j=1}^{k} c_{j}^{2}$ converges to $\|\mathbf{v}(\cdot, t)\|_{2 ; E}^{2}$ and $\operatorname{ess} \sup _{(0, T)}\|\mathbf{v}(\cdot, t)\|_{2 ; E} \leq \gamma$. To prove the second statement, fix $k \in \mathbb{N}$ and take first a function of the form $\varphi=\sum_{j=1}^{k} \varphi_{j} \mathbf{e}_{j}$. For such a function, by the othonormality of $\left(\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}, \ldots\right)$

$$
\int_{E}\left(\mathbf{v}_{n}-\mathbf{v}\right) \cdot \varphi d x=\sum_{j=1}^{k}\left(c_{n, j}-c_{j}\right) \varphi_{j} \rightarrow 0 \text { as } n \rightarrow \infty
$$

by the uniform convergence of $\left\{c_{n, j}\right\} \rightarrow c_{j}$ for $j=1, \ldots, k$. For a general $\varphi=\sum \varphi_{j} \mathbf{e}_{j} \in L^{2}\left(E ; \mathbb{R}^{3}\right)$, having fixed $\varepsilon>0$, there exists $k_{\varepsilon}$, depending on $\varepsilon$ and $\boldsymbol{\varphi}$, such that $\sum_{j>k_{\varepsilon}} \varphi_{j}^{2}<\varepsilon$. Then estimate

$$
\begin{aligned}
\left|\int_{E}\left[\mathbf{v}_{n}(t)-\mathbf{v}(t)\right] \cdot \boldsymbol{\varphi} d x\right| & \leq \sum_{j=1}^{k_{\varepsilon}}\left|c_{n, j}(t)-c_{j}(t)\right|\left|\varphi_{j}\right| \\
& +\underset{(0, T)}{\operatorname{esssup}}\left\|\mathbf{v}_{n}(t)-\mathbf{v}(t)\right\|_{2 ; E}\left(\sum_{j>k_{\varepsilon}} \varphi_{j}^{2}\right)^{\frac{1}{2}} .
\end{aligned}
$$

By the first of (9.4) a further subsequence out of $\left\{\mathbf{v}_{n}\right\}$ can be selected and relabeled with $n$, such that $\left\{\mathbf{v}_{n}\right\} \rightarrow \mathbf{v}^{\prime}$ and $\left\{\nabla \mathbf{v}_{n}\right\} \rightarrow \nabla \mathbf{w}$ weakly in $L^{2}\left(E_{T} ; \mathbb{R}^{3}\right)$. By the uniqueness of the weak limit $\mathbf{v}^{\prime}=\mathbf{v}$ and $\nabla \mathbf{w}=\nabla \mathbf{v}$. By the weak lower semicontinuity of the norm and the first of (9.4)

$$
\|\nabla \mathbf{v}\|_{2 ; E_{T}} \leq \liminf \left\|\nabla \mathbf{v}_{n}\right\|_{2 ; E_{T}} \leq \gamma
$$

Proposition $10.2\left\{\mathbf{v}_{n}\right\} \rightarrow \mathbf{v}$ strongly in $L^{2}\left(E_{T} ; \mathbb{R}^{3}\right)$.
The proof uses the following lemma
Lemma 10.1 (Friedrichs [14]) For every $\varepsilon>0$ there exist a positive integer $N_{\varepsilon}$ depending only on $\varepsilon$ and $|E|$, and independent of $\mathbf{v}_{n}$, and $N_{\varepsilon}$ linearly independent functions $\left\{\boldsymbol{\psi}_{\ell}\right\}_{\ell=1}^{N_{\varepsilon}} \subset L^{2}\left(E ; \mathbb{R}^{3}\right)$ such that

$$
\begin{equation*}
\left\|\mathbf{v}_{n}-\mathbf{v}\right\|_{2 ; E_{T}}^{2} \leq \sum_{\ell=1}^{N_{\varepsilon}} \int_{0}^{T}\left|\int_{E}\left(\mathbf{v}_{n}-\mathbf{v}\right) \cdot \boldsymbol{\psi}_{\ell} d x\right|^{2} d t+\varepsilon\left\|\nabla\left(\mathbf{v}_{n}-\mathbf{v}\right)\right\|_{2 ; E_{T}}^{2} \tag{10.1}
\end{equation*}
$$

Inequality (10.1) is a special case, applied to ( $\mathbf{v}_{n}-\mathbf{v}$ ) of a more general Friedrichs' Lemma, which we will prove in Section 10c of the Complements.

Proof (of Proposition 10.2). Fix $\varepsilon>0$ and determine $N_{\varepsilon}$ and the system $\left\{\boldsymbol{\psi}_{\ell}\right\}_{\ell=1}^{N_{\varepsilon}} \subset L^{2}\left(E ; \mathbb{R}^{3}\right)$. Let now $n \rightarrow \infty$ in (10.1). The first term goes to zero because of the weak uniform convergence of $\left(\mathbf{v}_{n}-\mathbf{v}\right)$ in $L^{2}\left(E ; \mathbb{R}^{3}\right)$. The last term is majorized by $2 \gamma^{2} \varepsilon$, where $\gamma$ is the constant in the first of (9.4).

## 11 The Limiting Process and Proof of Theorem 8.1

Let $\varphi_{k}=\sum_{\ell=1}^{k} \varphi_{\ell} \mathbf{e}_{\ell}$ for fixed $k \in \mathbb{N}$. Multiply (9.3) by $\varphi_{i}$, add for $i=1, \ldots, k$ and integrate over $(0, t) \subset(0, T)$ to obtain for $n \geq k$

$$
\begin{aligned}
\int_{E} \mathbf{v}_{n}(t) \cdot \boldsymbol{\varphi}_{k}(t) d x & -\int_{0}^{t} \int_{E} \mathbf{v}_{n} \cdot \boldsymbol{\varphi}_{k, \tau} d x d \tau \\
& +\nu \int_{0}^{t} \int_{E} \nabla \mathbf{v}_{n}: \nabla \boldsymbol{\varphi}_{k} d x d \tau \\
& +\int_{0}^{t} \int_{E}\left(\mathbf{v}_{n} \cdot \nabla\right) \mathbf{v}_{n} \cdot \boldsymbol{\varphi}_{k} d x d \tau \\
& =\int_{E} \mathbf{v}_{o} \cdot \boldsymbol{\varphi}_{k}(0) d x+\int_{0}^{t} \int_{E} \mathbf{f} \cdot \boldsymbol{\varphi}_{k} d x d \tau
\end{aligned}
$$

In turn, this is averaged in time over $(t, t+h) \subset(0, T)$, for a fixed $h>0$, sufficiently small so that $0<t+h<T$. Denoting by

$$
f_{t}^{t+h}\{\cdots\} d \tau=\frac{1}{h} \int_{t}^{t+h}\{\cdots\} d \tau
$$

such averages, gives

$$
\begin{aligned}
f_{t}^{t+h} \int_{E} \mathbf{v}_{n}(\tau) \cdot \boldsymbol{\varphi}_{k}(\tau) d x d \tau & -f_{t}^{t+h} \int_{0}^{\tau} \int_{E} \mathbf{v}_{n}(s) \cdot \boldsymbol{\varphi}_{k, s}(s) d x d s d \tau \\
& +\nu f_{t}^{t+h} \int_{0}^{\tau} \int_{E} \nabla \mathbf{v}_{n}(s): \nabla \boldsymbol{\varphi}_{k}(s) d x d s d \tau \\
& +f_{t}^{t+h} \int_{0}^{\tau} \int_{E}\left(\mathbf{v}_{n}(s) \cdot \nabla\right) \mathbf{v}_{n}(s) \cdot \boldsymbol{\varphi}_{k}(s) d x d s d \tau \\
= & \int_{E} \mathbf{v}_{o} \cdot \boldsymbol{\varphi}_{k}(0) d x+f_{t}^{t+h} \int_{0}^{\tau} \int_{E} \mathbf{f}(s) \cdot \boldsymbol{\varphi}_{k}(s) d x d s d \tau
\end{aligned}
$$

Let $n \rightarrow \infty$ by keeping $k$ fixed, to get

$$
\begin{aligned}
f_{t}^{t+h} \int_{E} \mathbf{v}(\tau) \cdot \boldsymbol{\varphi}_{k}(\tau) d x d \tau & -f_{t}^{t+h} \int_{0}^{\tau} \int_{E} \mathbf{v}(s) \cdot \boldsymbol{\varphi}_{k, s}(s) d x d s d \tau \\
& +\nu f_{t}^{t+h} \int_{0}^{\tau} \int_{E} \nabla \mathbf{v}(s): \nabla \boldsymbol{\varphi}_{k}(s) d x d s d \tau \\
& +f_{t}^{t+h} \int_{0}^{\tau} \int_{E}(\mathbf{v}(s) \cdot \nabla) \mathbf{v}(s) \cdot \boldsymbol{\varphi}_{k}(s) d x d s d \tau \\
& =\int_{E} \mathbf{v}_{o} \cdot \boldsymbol{\varphi}_{k}(0) d x+f_{t}^{t+h} \int_{0}^{\tau} \int_{E} \mathbf{f}(s) \cdot \boldsymbol{\varphi}_{k}(s) d x d s d \tau
\end{aligned}
$$

The various limits are justified by the weak convergence $\left\{\nabla \mathbf{v}_{n}\right\} \rightarrow \nabla \mathbf{v}$ and the strong convergence $\left\{\mathbf{v}_{n}\right\} \rightarrow \mathbf{v}$. In particular, such a strong convergence
permits one to pass to the limit in the non-linear term. In Section 11c of the Complements we will discuss a counterexample to show that in general, having weak convergence does not suffice to pass to the limit in such a term. Next, take $\varphi \in C^{\infty}(0, T ; \mathcal{V})$, write it as $\boldsymbol{\varphi}=\sum \varphi_{j} \mathbf{e}_{j}$, and let $\boldsymbol{\varphi}_{k}$ be its truncated series. Because of the predicated smoothness of $\varphi$

$$
\left\{\boldsymbol{\varphi}_{k}\right\},\left\{\nabla \boldsymbol{\varphi}_{k}\right\},\left\{\boldsymbol{\varphi}_{k, t}\right\} \rightarrow \boldsymbol{\varphi}, \nabla \boldsymbol{\varphi}, \boldsymbol{\varphi}_{t} \text { in } L^{2}\left(E_{T}\right)
$$

and also $\left\{\varphi_{k}\right\} \rightarrow \varphi$ in $L^{5}\left(E_{T} ; \mathbb{R}^{3}\right)$. Compute and estimate

$$
\begin{aligned}
& \limsup _{k \rightarrow \infty} \mid \iint_{E_{t}} \nabla \mathbf{v}: \nabla \boldsymbol{\varphi}_{k} d x d \tau-\iint_{E_{t}} \nabla \mathbf{v}: \nabla \boldsymbol{\varphi} d x d \tau \mid \\
& \leq\|\nabla \mathbf{v}\|_{2 ; E_{T}} \lim _{k \rightarrow \infty}\left\|\nabla\left(\boldsymbol{\varphi}_{k}-\boldsymbol{\varphi}\right)\right\|_{2 ; E_{T}}=0 .
\end{aligned}
$$

The limits in all the other terms, but the non-linear one, are treated similarly. For the non-linear term

$$
\begin{aligned}
& \limsup _{k \rightarrow \infty} \mid \iint_{E_{t}}(\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \boldsymbol{\varphi}_{k} d x d \tau-\iint_{E_{t}}(\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \boldsymbol{\varphi}_{k} d x d \tau \mid \\
& \leq\|\nabla \mathbf{v}\|_{2 ; E_{T}}\|\mathbf{v}\|_{\frac{10}{3} ; E_{T}} \lim _{k \rightarrow \infty}\left\|\boldsymbol{\varphi}_{k}-\boldsymbol{\varphi}\right\|_{5 ; E_{T}}
\end{aligned}
$$

Letting $k \rightarrow \infty$ yields, for all $\varphi \in C^{\infty}(0, T ; \mathcal{V})$

$$
\begin{aligned}
f_{t}^{t+h} \int_{E} \mathbf{v}(\tau) \cdot \boldsymbol{\varphi}(\tau) d x d \tau & -f_{t}^{t+h} \int_{0}^{\tau} \int_{E} \mathbf{v}(s) \cdot \boldsymbol{\varphi}_{s}(s) d x d s d \tau \\
& +\nu f_{t}^{t+h} \int_{0}^{\tau} \int_{E} \nabla \mathbf{v}(s): \nabla \boldsymbol{\varphi}(s) d x d s d \tau \\
& +f_{t}^{t+h} \int_{0}^{\tau} \int_{E}(\mathbf{v}(s) \cdot \nabla) \mathbf{v}(s) \cdot \boldsymbol{\varphi}(s) d x d s d \tau \\
& =\int_{E} \mathbf{v}_{o} \cdot \boldsymbol{\varphi}(0) d x+f_{t}^{t+h} \int_{0}^{\tau} \int_{E} \mathbf{f}(s) \cdot \boldsymbol{\varphi}(s) d x d s d \tau
\end{aligned}
$$

Finally let $h \rightarrow 0$ and notice that

$$
\lim _{h \rightarrow 0} f_{t}^{t+h} \int_{E} \mathbf{v}(\tau) \cdot \boldsymbol{\varphi}(\tau) d x d \tau=\int_{E} \mathbf{v}(t) \cdot \boldsymbol{\varphi}(t) d x \quad \text { for a.e. } t \in(0, T)
$$

since, for integrable functions in $(0, T)$, a.e. $t$ is a Lebesgue point. Thus, the function $\mathbf{v}$ so constructed satisfies the definition (8.2) of weak solution. It should be stressed that the testing functions $\varphi$ cannot, in general be taken out of $C^{1}(0, T ; V)$ as the limiting process for $k \rightarrow \infty$ requires a further smoothness, guaranteed in general by taking $\varphi \in C^{\infty}(0, T ; \mathcal{V})$.

## 12 Higher Integrability and Some Consequences

The Hopf solution has a limited degree of regularity due to the non-linear term $(\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \boldsymbol{\varphi}$. The weak formulation (8.2) holds for all $\boldsymbol{\varphi} \in C^{\infty}(0, T ; \mathcal{V}) \subset W$, whereas the solution $\mathbf{v}$ is required to be in $W$. If in (8.2) one could take $\boldsymbol{\varphi}=\mathbf{v}$ then, since $\operatorname{div} \mathbf{v}=0$, the non linear term would vanish and further regularity could be inferred on $\mathbf{v}$. Optimal local and global regularity of the Hopf solutions is unknown and it is a current major topic of investigation. To underscore this point, here we indicate some consequences of assuming higher integrability on $\mathbf{v}$ and on the various terms of (8.1), including the pressure term $\nabla p$.
Lemma 12.1 Let $\mathbf{v}$ be a Hopf solution of (8.1). Then $(\mathbf{v} \cdot \nabla) \mathbf{v} \in L^{\frac{5}{4}}\left(E_{T} ; \mathbb{R}^{3}\right)$, and

$$
\|(\mathbf{v} \cdot \nabla) \mathbf{v}\|_{\frac{5}{4} ; E_{T}} \leq\|\mathbf{v}\|_{\frac{10}{3} ; E_{T}}\|\nabla \mathbf{v}\|_{2 ; E_{T}}
$$

Proof. Let $q, q^{\prime}>1$ be Hölder conjugate and for $p>1$ to be chosen, compute and estimate

$$
\begin{equation*}
\iint_{E_{T}}|(\mathbf{v} \cdot \nabla) \mathbf{v}|^{p} d x d t \leq\left(\iint_{E_{T}}|\nabla \mathbf{v}|^{p q} d x d t\right)^{\frac{1}{q}}\left(\iint_{E_{T}}|\mathbf{v}|^{p q^{\prime}} d x d t\right)^{\frac{1}{q^{\prime}}} \tag{12.1}
\end{equation*}
$$

Choose $p q=2$ and $p q^{\prime}=\frac{10}{3}$ which yields $p=\frac{5}{4}$.
Assume momentarily that $\nabla p \in L_{\text {loc }}^{\frac{5}{4}}\left(E_{T} ; \mathbb{R}^{3}\right)$ and set

$$
\boldsymbol{\Phi}=\mathbf{f}-\nabla p-(\mathbf{v} \cdot \nabla) \mathbf{v} \in L_{\mathrm{loc}}^{\frac{5}{4}}\left(E_{T} ; \mathbb{R}^{3}\right)
$$

Then the weak formulation (8.2) yields ${ }^{4}$

$$
\begin{equation*}
\mathbf{v}_{t}-\nu \Delta \mathbf{v}=\boldsymbol{\Phi} \quad \text { weakly in } E_{T} \text { for all } \varphi \in C_{o}^{\infty}\left(E_{T} ; \mathbb{R}^{3}\right) \tag{12.2}
\end{equation*}
$$

This is a linear parabolic system with forcing term $\boldsymbol{\Phi} \in L_{\text {loc }}^{\frac{5}{4}}\left(E_{T} ; \mathbb{R}^{3}\right)$. Then by classical parabolic theory [11], the weak derivatives $\mathbf{v}_{x_{i} x_{j}}$ and $\mathbf{v}_{t}$ are in $L_{\text {loc }}^{\frac{5}{4}}\left(E_{T} ; \mathbb{R}^{3}\right)$. The argument can be repeated to yield further regularity on $\mathbf{v}$. Therefore, assuming a moderate degree of integrability of $\nabla p$ yields a considerably higher regularity on $\mathbf{v}$.

In $\S 20$, we will get back to the regularity of the pressure for Hopf solutions.

### 12.1 The $L^{p, q}\left(E_{T} ; \mathbb{R}^{N}\right)$ Spaces

For $p, q>1$ let

$$
L^{p, q}\left(E_{T} ; \mathbb{R}^{N}\right)=\left\{\begin{array}{c}
\text { Lebesgue measurable functions } \mathbf{f}: E_{T} \rightarrow \mathbb{R}^{N} \text { with } \\
\text { finite norm }\|\mathbf{f}\|_{p, q ; E_{T}}=\left(\int_{0}^{T}\|\mathbf{f}(\cdot, t)\|_{p ; E}^{q} d t\right)^{\frac{1}{q}}
\end{array}\right\}
$$

[^6]In what follows we let $p>N$ and $q>2$ be linked by

$$
\begin{equation*}
\frac{N}{p}+\frac{2}{q}=1 \tag{12.3}
\end{equation*}
$$

Condition (12.3) is known as the Ladyzhenskaya-Prodi-Serrin condition.
Recall also the following special case of the Gagliardo-Nirenberg embedding inequality ${ }^{5}$

$$
\|\mathbf{v}\|_{r ; E} \leq \gamma(N, p)\|\nabla \mathbf{v}\|_{2 ; E}^{\frac{N}{p}}\|\mathbf{v}\|_{2 ; E}^{\frac{2}{q}}, \quad \text { where } \quad r=\frac{2 p}{p-2}
$$

Lemma 12.2 There exists a constant $\gamma(N, p)$ depending only on $N$ and $p$, such that for any triple $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ with $\mathbf{u} \in L^{p, q}\left(E_{T} ; \mathbb{R}^{N}\right), \mathbf{v} \in W$, and $\mathbf{w} \in W$, there holds

$$
\begin{align*}
& \int_{0}^{T} \int_{E}|(\mathbf{v} \cdot \nabla) \mathbf{w} \cdot \mathbf{u}| d x d t \leq \gamma\|\mathbf{u}\|_{p, q ; E_{T}}\|\mathbf{v}\|_{W}\|\nabla \mathbf{w}\|_{2 ; E_{T}} ; \\
& \int_{0}^{T} \int_{E}|(\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{u}| d x d t  \tag{12.4}\\
& \quad \leq \gamma\left(\int_{0}^{T}\|\mathbf{u}(\cdot, t)\|_{p, E}^{q}\|\mathbf{w}(\cdot, t)\|_{2 ; E}^{2} d t\right)^{\frac{1}{q}}\|\nabla \mathbf{w}\|_{2 ; E_{T}}^{1+\frac{N}{p}}
\end{align*}
$$

Proof. By Hölder's inequality with conjugate exponents

$$
\frac{1}{r}+\frac{1}{p}=\frac{1}{2}, \quad \text { i.e., } \quad \frac{1}{r}+\frac{1}{p}+\frac{1}{2}=1
$$

using also the indicated special case of Gagliardo-Nirenberg inequality we have

$$
\begin{aligned}
\int_{E}|(\mathbf{v} \cdot \nabla) \mathbf{w} \cdot \mathbf{u}| d x & \leq\|\mathbf{v}\|_{r ; E}\|\nabla \mathbf{w}\|_{2 ; E}\|\mathbf{u}\|_{p ; E} \\
& \leq \gamma\|\mathbf{v}\|_{2 ; E}^{\frac{2}{q}}\|\nabla \mathbf{v}\|_{2 ; E}^{\frac{N}{p}}\|\nabla \mathbf{w}\|_{2 ; E}\|\mathbf{u}\|_{p ; E}
\end{aligned}
$$

Next integrate over $(0, T)$ and use Hölder's inequality with conjugate expo-

$$
\begin{aligned}
& \text { nents } \frac{N}{2 p}+\frac{1}{q}+\frac{1}{2}=1 \\
& \qquad \begin{array}{l}
\int_{0}^{T} \int_{E}|(\mathbf{v} \cdot \nabla) \mathbf{w} \cdot \mathbf{u}| d x d t \leq \gamma\left(\int_{0}^{T}\|\mathbf{v}(\cdot, t)\|_{2 ; E}^{2}\|\mathbf{u}(\cdot, t)\|_{p ; E}^{q} d t\right)^{\frac{1}{q}} \\
\times\left(\int_{0}^{T}\|\nabla \mathbf{v}(\cdot, t)\|_{2, E}^{2} d t\right)^{\frac{N}{2 p}}\left(\int_{0}^{T}\|\nabla \mathbf{w}(\cdot, t)\|_{2 ; E}^{2} d t\right)^{\frac{1}{2}} \\
\leq \gamma\left(\underset{(0, T)}{\operatorname{esssup}}\|\mathbf{v}(\cdot, t)\|_{2 ; E}\right)^{\frac{2}{q}}\|\nabla \mathbf{v}\|_{2 ; E_{T}}^{\frac{N}{p}}\|\nabla \mathbf{w}\|_{2 ; E_{T}}\|\mathbf{u}\|_{p, q ; E_{T}} \\
\leq \gamma\|\mathbf{v}\|_{W}\|\nabla \mathbf{w}\|_{2, E_{T}}\|\mathbf{u}\|_{p, q ; E_{T}} .
\end{array}
\end{aligned}
$$

This proves the first of (12.4). The proof of the second is the same by interchanging the roles of $\mathbf{v}$ and $\mathbf{w}$.

[^7]
### 12.2 The Case $N=2$

Lemma 12.3 Let $N=2$. Then for all $\mathbf{v} \in W$ (12.3) holds with $p=q=4$, and

$$
\|\mathbf{v}\|_{4 ; E_{T}} \leq \pi^{-\frac{1}{4}}\|\mathbf{v}\|_{W}
$$

Proof. The Gagliardo-Nirenberg multiplicative inequality for $u \in W_{o}^{1, p}(E)$ reads ${ }^{6}$

$$
\|u\|_{p^{*} ; E} \leq \gamma(N, p)\|\nabla u\|_{p ; E} \quad \text { where } \quad p^{*}=\frac{N p}{N-p} \quad \text { and } \quad 1 \leq p<N
$$

for a constant $\gamma(N, p)$ depending only on $N$ and $p$. When $p=1$ the optimal constant is $\gamma(N, 1)=\frac{1}{N}\left(\frac{N}{\omega_{N}}\right)^{\frac{1}{N}}$, where $\omega_{N}$ is the measure of the unit sphere in $\mathbb{R}^{N}$. Apply the inequality for $N=2$, with $u=|\mathbf{v}|^{2}$ and $p=1$ to get

$$
\begin{aligned}
\left(\int_{E}|\mathbf{v}|^{4} d x\right)^{\frac{1}{2}} & \left.\leq\left.\frac{1}{2} \frac{1}{\sqrt{\pi}} \int_{E}|\nabla| \mathbf{v}\right|^{2}\left|d x \leq \frac{1}{\sqrt{\pi}} \int_{E}\right| \mathbf{v}| | \nabla \mathbf{v} \right\rvert\, d x \\
& \leq \frac{1}{\sqrt{\pi}}\left(\int_{E}|\mathbf{v}|^{2} d x\right)^{\frac{1}{2}}\left(\int_{E}|\nabla \mathbf{v}|^{2} d x\right)^{\frac{1}{2}} \\
& \leq \frac{1}{\sqrt{\pi}}\left(\underset{(0, T)}{\operatorname{ess} \sup } \int_{E}|\mathbf{v}|^{2} d x\right)^{\frac{1}{2}}\left(\int_{E}|\nabla \mathbf{v}|^{2} d x\right)^{\frac{1}{2}}
\end{aligned}
$$

From this

$$
\int_{E}|\mathbf{v}(\cdot, t)|^{4} d x \leq \frac{1}{\pi}\|\mathbf{v}\|_{W}^{2} \int_{E}|\nabla \mathbf{v}(\cdot, t)|^{2} d x
$$

Integrating over $(0, T)$ yields

$$
\|\mathbf{v}\|_{4 ; E_{T}}^{4} \leq \frac{1}{\pi}\|\mathbf{v}\|_{W}^{2}\|\nabla \mathbf{v}\|_{2 ; E_{T}}^{2}
$$

Corollary 12.1 Any Hopf solution to (8.1) for $N=2$ satisfies (12.3) for $p=q=4$.

## 13 Energy Identity for the Homogeneous Boundary Value Problem with Higher Integrability

We get back to (8.1) with $\mathbf{f}=0$ to which we refer as the homogeneous problem and label it as $(8.1)_{o}$. A weak solution is meant in the sense of $(8.2)_{o}$, with $\mathbf{f}=0$, for all $\varphi \in C^{\infty}(0, T ; \mathcal{V})$. While a weak solution has been constructed by the Hopf's procedure we assume here that one is given and meant weakly.

[^8]Proposition 13.1 (Prodi [35]) Let $\mathbf{v}$ be a weak solution to (8.1) ${ }_{0}$. Moreover, assume that $\mathbf{v} \in L^{p, q}\left(E_{T} ; \mathbb{R}^{N}\right)$ with $p>N$ and $q>2$ satisfying (12.3). Then

$$
\begin{equation*}
\|\mathbf{v}(\cdot, t)\|_{2 ; E}^{2}+2 \nu\|\nabla \mathbf{v}\|_{2 ; E_{T}}^{2}=\left\|\mathbf{v}_{o}\right\|_{2 ; E}^{2} \quad \text { for a.e. } t \in(0, T) \tag{13.1}
\end{equation*}
$$

Proof. The proof consists in taking formally $\varphi=\mathbf{v}$ in (8.2) $)_{o}$. The assumption (12.3) makes this possible by a series of approximations. First, since $\mathbf{v} \in$ $L^{2}(0, T ; V)$ there exists a sequence $\left\{\mathbf{v}_{k}\right\} \subset C^{\infty}(0, T ; \mathcal{V})$ such that $\left\{\mathbf{v}_{k}\right\} \rightarrow \mathbf{v}$ in $L^{2}(0, T ; V)$. Next, let $J(\cdot)$ be the Friedrichs' mollifying kernel in $\mathbb{R}$ and denote by $J_{\varepsilon}(\cdot)$ its rescaled by a parameter $\varepsilon \in(0,1)$, i.e.

$$
J(\tau)=C\left\{\begin{array}{ll}
\exp \left(\frac{\tau^{2}}{\tau^{2}-1}\right) & \text { for }|\tau|<1, \\
0 & \text { for }|\tau| \geq 1,
\end{array} \quad J_{\varepsilon}(\tau)=\frac{1}{\varepsilon} J\left(\frac{\tau}{\varepsilon}\right)\right.
$$

where $C>0$ is a constant that normalizes the kernel $J$. Notice that

$$
J(-t)=J(t), \quad J^{\prime}(-t)=-J^{\prime}(t)
$$

Then for a.e. $t \in(0, T]$ fixed, set

$$
\begin{equation*}
\mathbf{v}_{\varepsilon, k}(\tau)=\int_{0}^{t} J_{\varepsilon}(\tau-s) \mathbf{v}_{k}(s) d s ; \quad \mathbf{v}_{\varepsilon}(\tau)=\int_{0}^{t} J_{\varepsilon}(\tau-s) \mathbf{v}(s) d s \tag{13.2}
\end{equation*}
$$

One verifies that $\mathbf{v}_{k, \varepsilon} \in C^{\infty}(0, T ; \mathcal{V})$ and therefore, it is an admissible test function in the weak formulation $(8.2)_{o}$. Such a choice gives

$$
\begin{aligned}
\int_{E} \mathbf{v}(t) \cdot \mathbf{v}_{\varepsilon, k}(t) d x & -\int_{0}^{t} \int_{E} \mathbf{v} \cdot \mathbf{v}_{\varepsilon, k ; \tau} d x d \tau \\
& +\int_{0}^{t} \int_{E}\left(\nu \nabla \mathbf{v}: \nabla \mathbf{v}_{\varepsilon, k}+(\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \mathbf{v}_{\varepsilon, k}\right) d x d \tau \\
& =\int_{E} \mathbf{v}_{o} \cdot \mathbf{v}_{\varepsilon, k}(0) d x
\end{aligned}
$$

Letting $k \rightarrow \infty$ now gives

$$
\begin{align*}
\int_{E} \mathbf{v}(t) \cdot \mathbf{v}_{\varepsilon}(t) d x & -\int_{0}^{t} \int_{E} \mathbf{v} \cdot \mathbf{v}_{\varepsilon ; \tau} d x d \tau \\
& +\int_{0}^{t} \int_{E}\left(\nu \nabla \mathbf{v}: \nabla \mathbf{v}_{\varepsilon}+(\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \mathbf{v}_{\varepsilon}\right) d x d \tau  \tag{13.3}\\
& =\int_{E} \mathbf{v}_{o} \cdot \mathbf{v}_{\varepsilon}(0) d x
\end{align*}
$$

The various limits, but the first one and the one regarding the non-linear term, are justified since $\left\{\mathbf{v}_{\varepsilon, k}\right\} \rightarrow \mathbf{v}_{\varepsilon}$ in $L^{2}(0, T ; V)$.

The limit of the first term is justified, for fixed $\varepsilon>0$ since $\left\{\mathbf{v}_{k}\right\} \rightarrow \mathbf{v}$ in $L^{2}\left(E_{T} ; \mathbb{R}^{N}\right)$ and the definition of $\mathbf{v}_{\varepsilon}$. Indeed,

$$
\begin{aligned}
\mid \int_{E} \mathbf{v}(t) \cdot\left[\mathbf{v}_{\varepsilon, k}(t)\right. & \left.-\mathbf{v}_{\varepsilon}(t)\right] d x \mid \\
& \leq \int_{E}|\mathbf{v}(t)| \int_{0}^{t} J_{\varepsilon}(t-s)\left|\mathbf{v}_{k}(s)-\mathbf{v}(s)\right| d s d x \\
& =\int_{0}^{t} J_{\varepsilon}(t-s)\left(\int_{E}\left|\mathbf{v}(t) \| \mathbf{v}_{k}(s)-\mathbf{v}(s)\right| d x\right) d s \\
& \leq \int_{0}^{t} J_{\varepsilon}(t-s)\|\mathbf{v}(t)\|_{2 ; E}\left\|\mathbf{v}_{k}(s)-\mathbf{v}(s)\right\|_{2 ; E} d s \\
& \leq \underset{(0, T)}{\operatorname{ess} \sup }\|\mathbf{v}(t)\|_{2 ; E} \int_{0}^{T} J_{\varepsilon}(t-s)\left\|\mathbf{v}_{k}(s)-\mathbf{v}(s)\right\|_{2 ; E} d s \\
& \leq\|\mathbf{v}\|_{W}\left(\int_{\mathbb{R}} J_{\varepsilon}^{2}(t) d t\right)^{\frac{1}{2}}\left\|\mathbf{v}_{k}-\mathbf{v}\right\|_{2 ; E_{T}}
\end{aligned}
$$

The last term tends to zero as $k \rightarrow \infty$ since $\left\{\mathbf{v}_{k}\right\} \rightarrow \mathbf{v}$ in $L^{2}\left(E_{T} ; \mathbb{R}^{N}\right)$. As for the non-linear term, compute and estimate

$$
\begin{aligned}
\left|\int_{0}^{t} \int_{E}(\mathbf{v} \cdot \nabla) \mathbf{v} \cdot\left(\mathbf{v}_{\varepsilon, k}-\mathbf{v}_{\varepsilon}\right) d x d \tau\right| & =\left|\int_{0}^{t} \int_{E}(\mathbf{v} \cdot \nabla)\left(\mathbf{v}_{\varepsilon, k}-\mathbf{v}_{\varepsilon}\right) \cdot \mathbf{v} d x d \tau\right| \\
& \leq \gamma\|\mathbf{v}\|_{W}\|\mathbf{v}\|_{p, q ; E_{T}}\left\|\nabla\left(\mathbf{v}_{\varepsilon, k}-\mathbf{v}_{\varepsilon}\right)\right\|_{2 ; E_{T}}
\end{aligned}
$$

by virtue of Lemma 12.2. This is indeed the role of the assumption (12.3) and the ensuing Lemma. The last term tends to zero as $k \rightarrow \infty$ since $\left\{\mathbf{v}_{\varepsilon, k}\right\} \rightarrow \mathbf{v}_{\varepsilon}$ in $L^{2}(0, T ; V)$.
Next, we let $\varepsilon \rightarrow 0$ in (13.3). For the first term we have

$$
\begin{aligned}
\int_{E} \mathbf{v}(t) \cdot \mathbf{v}_{\varepsilon}(t) d x= & \int_{E} \mathbf{v}(t) \int_{0}^{t} J_{\varepsilon}(t-s) \mathbf{v}(s) d s d x \\
= & \int_{E} \int_{0}^{t} J_{\varepsilon}(\eta) \mathbf{v}(t-\eta) \cdot \mathbf{v}(t) d \eta d x \\
= & \int_{E} \int_{0}^{t} J_{\varepsilon}(\eta)|\mathbf{v}(t)|^{2} d \eta d x \\
& +\int_{E} \int_{0}^{t} J_{\varepsilon}(\eta) \mathbf{v}(t) \cdot[\mathbf{v}(t-\eta)-\mathbf{v}(t)] d \eta d x
\end{aligned}
$$

Since $J_{\varepsilon}$ is even and it has been normalized, as $\varepsilon \rightarrow 0$,

$$
\int_{E} \int_{0}^{t} J_{\varepsilon}(\eta)|\mathbf{v}(t)|^{2} d \eta d x \rightarrow \frac{1}{2} \int_{E}|\mathbf{v}(t)|^{2} d x
$$

On the other hand

$$
\begin{aligned}
\mid \int_{0}^{t} J_{\varepsilon}(\eta) \int_{E} \mathbf{v}(t) \cdot & {[\mathbf{v}(t-\eta)-\mathbf{v}(t)] d x d \eta \mid } \\
& \leq \int_{0}^{t} J_{\varepsilon}(\eta)\left|\int_{E} \mathbf{v}(t) \cdot[\mathbf{v}(t-\eta)-\mathbf{v}(t)] d x\right| d \eta
\end{aligned}
$$

and the integral tends to zero as $|\eta|<\varepsilon \rightarrow 0$ by the weak continuity of $t \rightarrow \mathbf{v}(t)$ in $L^{2}(E)$. A similar result holds for the right-hand side of (13.3). The second term is identically zero in $\varepsilon$. Indeed, after interchanging the order of integration, it can be written as

$$
\int_{E}\left(\int_{0}^{t} \int_{0}^{t} J_{\varepsilon}^{\prime}(\tau-s) \mathbf{v}(s) \cdot \mathbf{v}(\tau) d s d \tau\right) d x
$$

Now the integral in $(\cdots)$, for a.e. fixed $x \in E$, is a double integral extended over the rectangle of vertices $\{(0,0),(t, 0),(t, t),(0, t)\}$, which, in turn is the union of two disjoint, equal triangles of vertices $\{(0,0),(t, 0),(t, t)\}$ and $\{(0,0),(t, t),(0, t)\}$. Now the argument $\mathbf{v}(s) \cdot \mathbf{v}(\tau)$ is even with respect to these triangles, whereas $J_{\varepsilon}^{\prime}(\tau-s)$ is odd.
Next,
$\left|\int_{0}^{t} \int_{E} \nabla \mathbf{v}: \nabla\left(\mathbf{v}_{\varepsilon}-\mathbf{v}\right) d x d \tau\right| \leq \int_{E} \int_{0}^{t}|\nabla \mathbf{v}| \int_{\mathbb{R}} J_{\varepsilon}(\tau-s)|\nabla[\mathbf{v}(s)-\mathbf{v}(\tau)]| d s d \tau d x$ and this tends to zero as $\varepsilon \rightarrow 0$. Finally, for the non-linear term compute and estimate, with the aid of Lemma 12.2,

$$
\begin{aligned}
& \int_{0}^{t} \int_{E}(\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \int_{0}^{t} J_{\varepsilon}(\tau-s)[\mathbf{v}(s)-\mathbf{v}(\tau)] d x d s d \tau \\
& \leq \gamma\|\mathbf{v}\|_{W}\|\mathbf{v}\|_{p, q ; E_{T}}\left(\int_{E} \int_{0}^{t}\left(\int_{0}^{t} J_{\varepsilon}(\tau-s)[\nabla \mathbf{v}(s)-\nabla \mathbf{v}(\tau)] d s\right)^{2} d \tau d x\right)^{\frac{1}{2}}
\end{aligned}
$$

which tends to zero as $\varepsilon \rightarrow 0$ by the property of the mollifiers. Observe that the limit of the non-linear term

$$
\lim _{\varepsilon \rightarrow 0} \int_{0}^{t} \int_{E}(\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \mathbf{v}_{\varepsilon} d x d \tau=\int_{0}^{t} \int_{E}(\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \mathbf{v} d x d \tau=0
$$

gives zero contribution since $\operatorname{div} \mathbf{v}=0$. Collecting these calculations proves (13.1).

Remark 13.1 For $N=2$ condition (12.3) is redundant, as already stated in Lemma 12.3.

## 14 Stability and Uniqueness for the Homogeneous Boundary Value Problem with Higher Integrability

Proposition 14.1 ([31]) Let $\mathbf{v}$ and $\mathbf{u}$ be two weak solutions of (8.1) with $\mathbf{f}=0$, originating from initial data $\mathbf{v}_{o}$ and $\mathbf{u}_{o}$ in $L^{2}\left(E ; \mathbb{R}^{N}\right)$, meant in the
sense of $(8.2)_{o}$, for all $\varphi \in C^{\infty}(0, T ; \mathcal{V})$. Moreover, assume that at least one $\mathbf{v}$ or $\mathbf{u}$, say for example $\mathbf{u}$ is in $L^{p, q}\left(E_{T} ; \mathbb{R}^{N}\right)$ with $p>N$ and $q>2$ satisfying (12.3). Assume finally that they both satisfy the energy estimates

$$
\begin{align*}
\|\mathbf{v}(\cdot, t)\|_{2 ; E}^{2}+2 \nu\|\nabla \mathbf{v}\|_{2 ; E_{t}}^{2} & \leq\left\|\mathbf{v}_{o}\right\|_{2 ; E}^{2}  \tag{14.1}\\
\|\mathbf{u}(\cdot, t)\|_{2 ; E}^{2}+2 \nu\|\nabla \mathbf{u}\|_{2 ; E_{T}}^{2} & \leq\left\|\mathbf{u}_{o}\right\|_{2 ; E}^{2}
\end{align*} \quad \text { for a.e. } t \in(0, T) .
$$

Then, there exist a constant $\gamma$ depending only upon $N$ and $\nu$ such that setting $\mathbf{w}=\mathbf{v}-\mathbf{u}$ there holds

$$
\|\mathbf{w}(\cdot, t)\|_{2 ; E}^{2} \leq\left\|\mathbf{w}_{o}\right\|_{2 ; E}^{2} \exp \left\{\gamma \int_{0}^{t}\|\mathbf{u}(\cdot, \tau)\|_{p ; E}^{q} d \tau\right\}
$$

for a.e. $t \in(0, T)$.
Remark 14.1 If both $\mathbf{v}$ and $\mathbf{u}$ are in $L^{p, q}\left(E_{T} ; \mathbb{R}^{N}\right)$ with $p>N$ and $q>2$ satisfying (12.3) then by Proposition 13.1, the energy estimates (14.1) are satisfied. The Proposition is a statement of stability and uniqueness. If $N=2$, $\mathbf{v}$ and $\mathbf{u}$ are both in $L^{4}\left(E_{T} ; \mathbb{R}^{2}\right)$ and therefore, weak solutions are unique.

Proof. Let $\mathbf{v}$ and $\mathbf{u}$ be two weak solutions to (8.1) originating from initial data $\mathbf{v}_{o}$ and $\mathbf{u}_{o}$ in $L^{2}(E)$, meant in the sense of $(8.2)_{o}$, with $\mathbf{f}=0$, for all $\boldsymbol{\varphi} \in C^{\infty}(0, T ; \mathcal{V})$. In the weak formulation of $\mathbf{v}$ take the testing function $\mathbf{u}_{\varepsilon, k}$ defined as in (13.2) and in the weak formulation of $\mathbf{u}$ take the testing function $\mathbf{v}_{\varepsilon, k}$. Then let $k \rightarrow \infty$ by the same arguments as in the proof of Proposition 13.1, and add the resulting identities getting

$$
\begin{aligned}
\int_{E} & {\left[\mathbf{v}(t) \cdot \mathbf{u}_{\varepsilon}(t)+\mathbf{v}_{\varepsilon}(t) \cdot \mathbf{u}(t)\right] d x } \\
& -\int_{E}\left(\int_{0}^{t} \int_{0}^{t} J_{\varepsilon}^{\prime}(\tau-s)[\mathbf{v}(\tau) \cdot \mathbf{u}(s)+\mathbf{v}(s) \cdot \mathbf{u}(\tau)] d s d \tau\right) d x \\
& +\nu \int_{0}^{t} \int_{E}\left(\nabla \mathbf{v}: \nabla \mathbf{u}_{\varepsilon}+\nabla \mathbf{v}_{\varepsilon}: \nabla \mathbf{u}\right) d x d \tau \\
& +\int_{0}^{t} \int_{E}\left[(\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \mathbf{u}_{\varepsilon}+(\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v}_{\varepsilon}\right] d x d \tau \\
& =\int_{E}\left[\mathbf{v}_{o} \cdot \mathbf{u}_{\varepsilon}(0)+\mathbf{v}_{\varepsilon}(0) \cdot \mathbf{u}_{o}\right] d x
\end{aligned}
$$

Arguing as in the proof of Proposition 13.1, the second integral is identically zero in $\varepsilon$ since the argument $[\mathbf{v}(\tau) \mathbf{u}(s)+\mathbf{v}(s) \mathbf{u}(\tau)]$ is even with respect to the two triangles of vertices $\{(0,0),(t, 0),(t, t)\}$ and $\{(0,0),(t, t),(0, t)\}$ and $J_{\varepsilon}^{\prime}$ is odd with respect to the same triangles. We may now let $\varepsilon \rightarrow$ by the same arguments and get

$$
\begin{align*}
\int_{E} \mathbf{v}(\cdot, t) \cdot \mathbf{u}(\cdot, t) d x & +2 \nu \int_{0}^{t} \int_{E} \nabla \mathbf{v}: \nabla \mathbf{u} d x d \tau \\
& +\int_{0}^{t} \int_{E}[(\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \mathbf{u}+(\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v}] d x d \tau  \tag{14.2}\\
& =\int_{E} \mathbf{v}_{o} \cdot \mathbf{u}_{o} d x
\end{align*}
$$

Next observe that since weak solutions are divergence free

$$
\begin{aligned}
& \int_{0}^{t} \int_{E}(\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \mathbf{u} d x d \tau=-\int_{0}^{t} \int_{E}(\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} d x d \tau \\
& \int_{0}^{t} \int_{E}(\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} d x d \tau=\int_{0}^{t} \int_{E}(\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{w} d x d \tau
\end{aligned}
$$

where we have set $\mathbf{w}=\mathbf{v}-\mathbf{u}$. Using again that $\mathbf{w}$ is divergence free, the sum of these terms equals

$$
\int_{0}^{t} \int_{E}[(\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \mathbf{u}+(\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v}] d x d \tau=-\int_{0}^{t} \int_{E}(\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{u} d x d \tau
$$

Adding the energy inequalities (14.1) and subtracting (14.2) multiplied by 2 gives

$$
\|\mathbf{w}(t)\|_{2 ; E}^{2}+2 \nu\|\nabla \mathbf{w}\|_{2 ; E_{t}}^{2} \leq\left\|\mathbf{w}_{o}\right\|_{2 ; E}^{2}+\left|\int_{0}^{t} \int_{E}(\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{u} d x d \tau\right|
$$

The right hand side is estimated by the second of (12.4) of Lemma 12.2, and Young's inequality with conjugate exponents $\frac{1}{q}$ and $\frac{1}{2}+\frac{N}{2 p}$, and gives

$$
\left|\int_{0}^{t} \int_{E}(\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{u} d x d \tau\right| \leq \gamma \int_{0}^{t}\|\mathbf{u}(\tau)\|_{p ; E}^{q}\|\mathbf{w}(\tau)\|_{2 ; E}^{2} d \tau+2 \nu\|\nabla \mathbf{w}\|_{2 ; E_{t}}^{2}
$$

for a constant $\gamma$ depending only upon $N$ and $\nu$. Combining these estimates gives

$$
\|\mathbf{w}(t)\|_{2 ; E}^{2} \leq\left\|\mathbf{w}_{o}\right\|_{2 ; E}^{2}+\gamma \int_{0}^{t}\|\mathbf{u}(\tau)\|_{p ; E}^{q}\|\mathbf{w}(\tau)\|_{2 ; E}^{2} d \tau
$$

The proof is concluded by an application of Gronwall's inequality.

## 15 Local Regularity of Solutions with Higher Integrability

We continue assuming the higher integrability (12.3), and we address the smoothness of weak solutions. Notice that there is a difference between studying the regularity of solutions to the initial-boundary value problem (8.1) or the interior regularity.

In this second case, we deal with the intrinsic properties of the NavierStokes equations; indeed, we consider a local, weak solution in $E_{T}$, namely $\mathbf{v}$ which is weakly divergence free in $E_{T}$, and such that for any subset

$$
\Omega_{t_{1}, t_{2}} \stackrel{\text { def }}{=} \Omega \times\left(t_{1}, t_{2}\right) \subset \subset E_{T}
$$

we have

$$
\mathbf{v} \in L^{2}\left(t_{1}, t_{2} ; W^{1,2}(\Omega)\right) \cap L^{\infty}\left(t_{1}, t_{2} ; L^{2}(\Omega)\right)
$$

and $\mathbf{v}$ satisfies (8.2) for all soleinodal test functions $\boldsymbol{\varphi} \in C_{o}^{\infty}\left(\Omega_{t_{1}, t_{2}}\right)$.
If we consider a function $\psi=\psi(x)$ harmonic in $\Omega$ and an integrable function $a=a(t)$, it is a matter of straightforward computation to check that

$$
\mathbf{v}=a(t) \nabla \psi(x)
$$

is a local, weak solution of the Navier-Stokes equations for $\mathbf{f}=0$. Hence, it is infinitely differentiable with respect to space, but it might have integrable singularities with respect to time.

This example, which is due to Serrin ([40]), suggests that the time differentiability of a weak solution is directly connected to the time regularity which is assumed from the very beginning.

Moreover, as pointed out by Galdi (see [17, Page 41]), these highly irregular solutions exist because possible singularities are absorbed by the pressure term. As summarized by Struwe in [49], local regularity properties are not influenced by the nonlocal effects of the pressure, as long as we are interested only in boundedness and spatial regularity.

The situation is different if one considers the initial-boundary value problem (8.1) and its weak formulation (8.2), where one can hope to gain regularity in time from the assigned conditions. This has to do with the incompressibility of the fluids, since a sudden modification of the boundary value of the motion will be immediately felt throughout the whole flow region.

In this section we report a sufficient condition for the interior regularity, whereas in a subsequent section we will get back to regularity for the initialboundary value problem.
Theorem 15.1 ([40]). Let $\mathbf{v}$ be a local, weak solution of the Navier-Stokes equations in $E_{T}$ in the sense defined before.

Assume that $\mathbf{f}$ is conservative and at least in $L^{1,1}\left(E_{T} ; \mathbb{R}^{N}\right)$, and that $\mathbf{v} \in$ $L^{p, q}\left(E_{T} ; \mathbb{R}^{N}\right)$ where $p>N, q>2$ satisfy (12.3). Then $\mathbf{v}$ is of class $C^{\infty}$ with respect to the space variable $x$, and each space derivative is bounded in compact subsets of $E_{T}$.

If, in addition, $\mathbf{v}_{t} \in L^{2, s}\left(E_{T} ; \mathbb{R}^{N}\right)$ for some $s \geq 1$, then the space derivatives are absolutely continuous functions of time. Moreover, there exists a strongly differentiable function $p=p(x, t)$ such that

$$
\begin{equation*}
\mathbf{v}_{t}-\nu \Delta \mathbf{v}+(\mathbf{v} \cdot \nabla) \mathbf{v}+\nabla p=\mathbf{f} \tag{15.1}
\end{equation*}
$$

almost everywhere in $E_{T}$.

Remark 15.1 Due to the local nature of the result, without loss of generality, one could more generally assume that $\mathbf{v} \in L^{p, q}\left(\mathcal{K} ; \mathbb{R}^{N}\right)$ for any $\mathcal{K} \subset \subset E_{T}$, with similar local integrability assumptions on $\mathbf{f}$ and $\mathbf{v}_{t}$.

Remark 15.2 If we limit ourselves to $N=3$, using the Sobolev inequalities, one can show that a weak solution naturally belongs to $L^{p, q}\left(E_{T} ; \mathbb{R}^{N}\right)$ where $\frac{3}{p}+\frac{2}{q} \geq \frac{3}{2}$ (for a brief discussion of this fact, see [24, Section 1]); hence, there is a gap between the natural regularity of $\mathbf{v}$ and what is assumed in (12.3) in order to have differentiability in space for $\mathbf{v}$.
Theorem 15.1 is originally due to Serrin ([40]), who developed some of the methods introduced by [33] few years before. Moreover, he used the stronger condition

$$
\begin{equation*}
\frac{N}{p}+\frac{2}{q}<1 \tag{15.2}
\end{equation*}
$$

The full (12.3) with $p>N$ was proved by [12, 44, 49]; see also [18]. The limiting case of $p=N$ was dealt with in [49] under a smallness condition; namely, Struwe assumes that $\mathbf{v} \in L^{N, \infty}\left(E_{T} ; \mathbb{R}^{N}\right)$ and that there is a $\rho>0$ such that

$$
\begin{equation*}
\int_{B_{\rho} \cap E}|\mathbf{v}(\cdot, t)|^{N} d x \leq \epsilon \tag{15.3}
\end{equation*}
$$

uniformly with respect to $t$ in $(0, T)$ for some absolute constant $\epsilon$ (see also [44]).

For $N=3$, condition (15.3) was fully removed in [8]. The regularity approach to $L^{3, \infty}$-solutions developed in [8] requires a completely different method with respect to Serrin's techniques and further developments, and the proof is based on the reduction of the regularity problem to a backward uniqueness problem.

For the sake of simplicity, here we present the original proof of [40], and therefore, we limit ourselves to (15.2).

At the end of [40], Serrin conjectures that under the same assumptions on $\mathbf{v}$ and $\mathbf{f}$, it should be possible to prove that solutions are analytic in the space variables. This was indeed proved by Kahane (see [21]).

Let $V^{2}$ be the closure of $\mathcal{V}$ in $W^{2,2}(E)$ : for $N=2$ and $N=3$ and the initial datum $\mathbf{v}_{o} \in V^{2}$, Kiselev and Ladyzhenskaya (see [22]) have proved the existence of a weak solution of the initial-boundary value problem (8.1) with

$$
\mathbf{v} \in L^{4, \infty}\left(E_{T} ; \mathbb{R}^{N}\right), \quad \mathbf{v}, \nabla \mathbf{v}, \mathbf{v}_{t} \in L^{2, \infty}\left(E_{T} ; \mathbb{R}^{N}\right)
$$

hence, Theorem 15.1 contains as a special case that the Kiselev-Ladyzhenskaya solution is of class $C^{\infty}$ in the space variable, and is Lipschitz continuous in time, at least if $\mathbf{f}$ is conservative.

Moreover, if $N=2$ or $N=3$ and the initial data are smooth enough for the Kiselev-Ladyzhenskaya solution to exist, then by Proposition 14.1, Hopf's solution must be the same and consequently has to be of class $C^{\infty}$ in the space variables.

Remark 15.3 As pointed out by Serrin in [41, p. 76], for the KiselevLadyzhenskaya solutions, the case $N=4$ can be treated by methods similar to the ones employed by the authors in [22].

## 16 Proof of Theorem 15.1- Introductory Results

In the following, $\Omega_{t_{1}, t_{2}}=\Omega \times\left(t_{1}, t_{2}\right)$ will denote an open set compactly contained in $E_{T}$; moreover, we will frequently deal with convolution integrals of the type

$$
h(x, t)=\iint_{\Omega_{t_{1}, t_{2}}} k(x-\xi, t-\tau) g(\xi, \tau) d \xi d \tau
$$

and we will write $h(x, t)=(k * g)(x, t)$. A first fundamental result is given by Proposition 16.1 Let $k \in L^{p, p^{\prime}}\left(\mathbb{R}^{N} \times \mathbb{R} ; \mathbb{R}\right)$ and $g \in L^{q, q^{\prime}}\left(\Omega_{t_{1}, t_{2}} ; \mathbb{R}\right)$ with $N \geq 1$, and

$$
\begin{equation*}
\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1, \quad \frac{1}{p^{\prime}}+\frac{1}{q^{\prime}}=\frac{1}{r^{\prime}}+1 \tag{16.1}
\end{equation*}
$$

Then, for the convolution

$$
h(x, t) \stackrel{\text { def }}{=} \iint_{\Omega_{t_{1}, t_{2}}} k(x-\xi, t-\tau) g(\xi, \tau) d \xi d \tau, \quad(x, t) \in \Omega_{t_{1}, t_{2}}
$$

we have

$$
\|h\|_{r, r^{\prime}} \leq\|k\|_{p, p^{\prime}}\|g\|_{q, q^{\prime}} .
$$

For the proof, we refer to Section 16c of the Complements.
We will take as $k=k(x, t)$ a space derivative of the fundamental solution $\Gamma$ of the heat equation

$$
\begin{equation*}
\frac{\partial u}{\partial t}-\nu \Delta u=0 \tag{16.2}
\end{equation*}
$$

It is usually considered in $\mathbb{R}^{N} \times(0,+\infty)$, and here we extend it to $\mathbb{R}^{N} \times \mathbb{R}$, also taking into account a general diffusion coefficient $\nu>0$, not necessarily equal to 1 ; we set

$$
\Gamma(x, t)=\left\{\begin{array}{rl}
\frac{1}{(4 \pi \nu t)^{\frac{N}{2}}} \exp \left(-\frac{|x|^{2}}{4 \nu t}\right) & t>0  \tag{16.3}\\
0 & t \leq 0
\end{array}\right.
$$

We have
Lemma 16.1 Let $k$ be a space derivative of the function $\Gamma$ defined in (16.3). Then for any $g \in L^{q, q^{\prime}}\left(\Omega_{t_{1}, t_{2}} ; \mathbb{R}\right)$, given $h=(k * g)(x, t)$ we have

$$
\|h\|_{r, r^{\prime} ; \Omega_{t_{1}, t_{2}}} \leq \gamma\|g\|_{q, q^{\prime} ; \Omega_{t_{1}, t_{2}}}
$$

where $\gamma=\gamma\left(t_{2}-t_{1}, \nu, N, q, q^{\prime}, r, r^{\prime}\right)$, provided that $1 \leq q \leq r, 1 \leq q^{\prime} \leq r^{\prime}$, and $N\left(\frac{1}{q}-\frac{1}{r}\right)+2\left(\frac{1}{q^{\prime}}-\frac{1}{r^{\prime}}\right)<1$.

Proof. Since $k=\frac{\partial \Gamma}{\partial x_{i}}$ with $i=1, \ldots, N$, we have

$$
\begin{equation*}
|k(x, t)| \leq \gamma_{1} t^{-\frac{N}{2}-1}|x| \exp \left(-\frac{|x|^{2}}{4 \nu t}\right) \tag{16.4}
\end{equation*}
$$

where $\gamma_{1}=\gamma_{1}(\nu, N)$. Moreover, since both $t$ and $\tau$ belong to $\left(t_{1}, t_{2}\right)$, taking into account the definition of $\Gamma$, we have

$$
\|k\|_{p, p^{\prime}} \leq\left(\int_{0}^{t_{2}-t_{1}}\left(\int_{\mathbb{R}^{N}}|k|^{p} d x\right)^{p^{\prime} / p} d t\right)^{1 / p^{\prime}}
$$

Taking (16.4) into account, we have

$$
\begin{aligned}
\left(\int_{\mathbb{R}^{N}}|k|^{p} d x\right)^{1 / p} & =\frac{\gamma_{1}}{t^{\frac{N}{2}+1}}\left(\int_{\mathbb{R}^{N}}|x|^{p} \exp \left(-p \frac{|x|^{2}}{4 \nu t}\right) d x\right)^{1 / p} \\
& =\frac{\gamma_{2}}{t^{\frac{N}{2}+1}} t^{\frac{N+p}{2 p}}\left(\int_{0}^{+\infty} s^{N+p-1} \exp \left(-\frac{p}{4} s^{2}\right) d s\right)^{1 / p} \\
& =\gamma_{2} t^{-\alpha}
\end{aligned}
$$

where $\gamma_{2}=\gamma_{2}(\nu, N, p)$ and $\alpha=\frac{N}{2}\left(1-\frac{1}{p}\right)+\frac{1}{2}$. Hence,

$$
\|k\|_{p, p^{\prime}} \leq\left(\int_{0}^{t_{2}-t_{1}} \gamma_{2}^{p^{\prime}} t^{-\alpha p^{\prime}} d t\right)^{1 / p^{\prime}}=\gamma_{3}\left(t_{2}-t_{1}\right)^{-\alpha+\frac{1}{p^{\prime}}}
$$

provided that $\alpha p^{\prime}<1$, and where $\gamma_{3}=\gamma_{3}\left(\nu, N, p, p^{\prime}\right)$. From (16.1) we have that

$$
1-\frac{1}{p}=\frac{1}{q}-\frac{1}{r}, \quad \frac{1}{p^{\prime}}=\frac{1}{r^{\prime}}-\frac{1}{q^{\prime}}+1
$$

hence, condition $\alpha<\frac{1}{p^{\prime}}$ can be rewritten as

$$
\frac{N}{2}\left(\frac{1}{q}-\frac{1}{r}\right)+\frac{1}{2}<\frac{1}{r^{\prime}}-\frac{1}{q^{\prime}}+1 \quad \Rightarrow \quad N\left(\frac{1}{q}-\frac{1}{r}\right)+2\left(\frac{1}{q^{\prime}}-\frac{1}{r^{\prime}}\right)<1
$$

In the sequel, we will work with $\boldsymbol{\omega}$, the so-called vorticity of the fluid; we have

$$
\begin{array}{ll}
N=2 & \boldsymbol{\omega}=\operatorname{curl} \mathbf{v}=\left(0,0, \frac{\partial v_{2}}{\partial x_{1}}-\frac{\partial v_{1}}{\partial x_{2}}\right) \\
N=3 & \boldsymbol{\omega}=\operatorname{curl} \mathbf{v}=\left(\frac{\partial v_{3}}{\partial x_{2}}-\frac{\partial v_{2}}{\partial x_{3}}, \frac{\partial v_{1}}{\partial x_{3}}-\frac{\partial v_{3}}{\partial x_{1}}, \frac{\partial v_{2}}{\partial x_{1}}-\frac{\partial v_{1}}{\partial x_{2}}\right)
\end{array}
$$

when $N>3, \boldsymbol{\omega}$ is an $(N-2)$-skew symmetric tensor, whose components are

$$
\omega_{k l}=\frac{\partial v_{k}}{\partial x_{l}}-\frac{\partial v_{l}}{\partial x_{k}} .
$$

Remark 16.1 In order to streamline the presentation and avoid distinctions for the values of $N$, in the following we will always write $\boldsymbol{\omega}=\operatorname{curl} \mathbf{v}$ and on the other hand, even when $N=2$ or $N=3$, we will think of $\boldsymbol{\omega}$ as a skew symmetric tensor; for example, for $N=3$, we will have

$$
\boldsymbol{\omega}=\left[\begin{array}{ccc}
0 & \frac{\partial v_{1}}{\partial x_{2}}-\frac{\partial v_{2}}{\partial x_{1}} & \frac{\partial v_{1}}{\partial x_{3}}-\frac{\partial v_{3}}{\partial x_{1}} \\
\frac{\partial v_{2}}{\partial x_{1}}-\frac{\partial v_{1}}{\partial x_{2}} & 0 & \frac{\partial v_{2}}{\partial x_{3}}-\frac{\partial v_{3}}{\partial x_{2}} \\
\frac{\partial v_{3}}{\partial x_{1}}-\frac{\partial v_{1}}{\partial x_{3}} & \frac{\partial v_{3}}{\partial x_{2}}-\frac{\partial v_{2}}{\partial x_{3}} & 0
\end{array}\right]
$$

and similarly for $N=2$.
Let $\boldsymbol{A}=\left(A_{1}, A_{2}, \ldots, A_{N}\right)$ : we define

$$
\boldsymbol{A} \wedge \boldsymbol{\omega}=\boldsymbol{B} \equiv\left(B_{1}, B_{2}, \cdots, B_{N}\right)
$$

where

$$
\begin{equation*}
B_{k}=\sum_{l=1}^{N} A_{l} \omega_{k l}=\sum_{l=1}^{N} A_{l}\left(\frac{\partial v_{k}}{\partial x_{l}}-\frac{\partial v_{l}}{\partial x_{k}}\right) \tag{16.5}
\end{equation*}
$$

## 17 Proof of Theorem 15.1 Continued

Let $E$ be a region in $\mathbb{R}^{N}, N \geq 2$ and $\Omega \subset E$ an open set such that $\bar{\Omega}$ is compact in $E$. Let $\mathbf{v} \in V$ (i.e. $|\nabla \mathbf{v}| \in L^{2}(\Omega)$, $\operatorname{div} \mathbf{v}=0$ weakly) and consider the vorticity $\boldsymbol{\omega}=$ curl $\mathbf{v}$, where we take into account the previous definition and Remark 16.1.

Theorem 17.1. Let $y \in \Omega$; then there exists a vector $\boldsymbol{A}=\boldsymbol{A}(y)$, harmonic in $\Omega$, such that

$$
\mathbf{v}(y)=\int_{\Omega} \nabla_{x} H(y-x) \wedge \boldsymbol{\omega}(x) d x+\boldsymbol{A}(y)
$$

where $H(y-x)$ is the fundamental solution of the Laplacean in $\mathbb{R}^{N}$ centered at $y$.

Proof. Let $x \mapsto u(x)$ be a $C_{o}^{\infty}\left(\mathbb{R}^{N}\right)$ scalar function and $H(y-x)$ be the solution of

$$
-\Delta H(y-x)=\delta_{y} \quad\left(\delta_{y} \quad \text { Dirac mass at } y\right)
$$

in $\mathcal{D}^{\prime}\left(\mathbb{R}^{N}\right)$. Since $u \in C_{o}^{\infty}\left(\mathbb{R}^{N}\right)$, in the sense of distributions we have

$$
\langle-\Delta H(y-x), u\rangle=\left\langle\delta_{y}, u(x)\right\rangle=u(y) \quad \Rightarrow \quad u(y)=\langle H(y-x),-\Delta u(x)\rangle
$$

Since $-\Delta u(x) \in C_{o}^{\infty}\left(\mathbb{R}^{N}\right)$ and $H(y-x)$ is summable,

$$
\begin{aligned}
u(y) & =\langle H(y-x),-\Delta u(x)\rangle=-\int_{\mathbb{R}^{N}} \Delta u(x) H(y-x) d x= \\
& =-\int_{\Omega} \Delta u(x) H(y-x) d x-\int_{\mathbb{R}^{N} \backslash \Omega} \Delta u(x) H(y-x) d x \\
& =I_{1}+I_{2} .
\end{aligned}
$$

Let us start with the computation of $I_{2}$. Over $\mathbb{R}^{N} \backslash \Omega$ we have $H(y-x) \in C^{\infty}$. Hence,

$$
\begin{aligned}
-\int_{\mathbb{R}^{N} \backslash \Omega} \Delta u(x) H(y-x) d x= & \int_{\partial \Omega} H(y-x) \nabla u(x) \cdot \mathbf{n} d \sigma \\
& -\int_{\mathbb{R}^{N} \backslash \Omega} \nabla u(x) \cdot \nabla H(y-x) d x=J_{1}+J_{2}
\end{aligned}
$$

Moreover,

$$
J_{2}=\int_{\partial \Omega} u(x) \nabla H(y-x) \cdot \mathbf{n} d \sigma+\underbrace{\int_{\mathbb{R}^{N} \backslash \Omega} u(x) \Delta H(y-x) d x}_{=0} .
$$

Hence,

$$
I_{2}=\int_{\partial \Omega} H(y-x) \nabla u(x) \cdot \mathbf{n} d \sigma+\int_{\partial \Omega} u(x) \nabla H(y-x) \cdot \mathbf{n} d \sigma
$$

Coming to the computation of $I_{1}$, since $u \in C_{o}^{\infty}\left(\mathbb{R}^{N}\right)$, we have

$$
\begin{aligned}
I_{1}= & -\int_{\Omega} \Delta u(x) H(y-x) d x=-\int_{\partial \Omega} H(y-x) \nabla u(x) \cdot \mathbf{n} d \sigma+ \\
& +\int_{\Omega} \nabla u(x) \cdot \nabla H(y-x) d x .
\end{aligned}
$$

Finally, summing up

$$
\begin{align*}
u(y)= & \int_{\Omega} \nabla u(x) \cdot \nabla H(y-x) d x  \tag{17.1}\\
& +\int_{\partial \Omega} u(x) \nabla H(y-x) \cdot \mathbf{n} d \sigma .
\end{align*}
$$

Notice that up to now we have assumed $u \in C_{o}^{\infty}$. However, a careful inspection of the proof shows that in (17.1) the only requirement to the existence of the integrals is $u \in W^{1,2}(\Omega)$, so that $u$ has $L^{2}$ trace over $\partial \Omega$, where $\partial \Omega$ is assumed smooth. Therefore, by a limiting process and a standard approximation procedure, we have that for any $u \in W^{1,2}(\Omega)$

$$
u(y)=\int_{\Omega} \nabla H(y-x) \cdot \nabla u(x) d x+\int_{\partial \Omega} u(x) \nabla H(y-x) \cdot \mathbf{n} d \sigma
$$

for a.e. $y \in \Omega$.
Let now $\mathbf{v} \in V$ with $\mathbf{v}=\left(v_{1}, v_{2}, \ldots, v_{N}\right)$. For each $k=1,2, \ldots, N$ and for a.e. $y \in \Omega$

$$
v_{k}(y)=\int_{\Omega} \nabla H(y-x) \cdot \nabla v_{k}(x) d x+\int_{\partial \Omega} v_{k}(x) \nabla H(y-x) \cdot \mathbf{n} d \sigma .
$$

We rewrite

$$
\begin{aligned}
\int_{\Omega} \nabla H(y-x) \cdot \nabla v_{k}(x) d x= & \int_{\Omega} \sum_{i=1}^{N} \frac{\partial H(y-x)}{\partial x_{i}}\left(\frac{\partial v_{k}(x)}{\partial x_{i}}-\frac{\partial v_{i}(x)}{\partial x_{k}}\right) d x \\
& +\int_{\Omega} \sum_{i=1}^{N} \frac{\partial H(y-x)}{\partial x_{i}} \frac{\partial v_{i}(x)}{\partial x_{k}} d x=\Sigma_{1}+\Sigma_{2} .
\end{aligned}
$$

By (16.5)

$$
\Sigma_{1}=\int_{\Omega}(\nabla H(y-x) \wedge \operatorname{curl} \mathbf{v}(x))_{k} d x=\int_{\Omega}(\nabla H(y-x) \wedge \boldsymbol{\omega}(x))_{k} d x
$$

and

$$
\Sigma_{2}=\int_{\partial \Omega} H(y-x) \frac{\partial}{\partial x_{k}} \sum_{i=1}^{N}\left(v_{i} n_{i}\right)(x) d \sigma-\int_{\Omega} H(y-x) \frac{\partial}{\partial x_{k}} \sum_{i=1}^{N} \frac{\partial v_{i}(x)}{\partial x_{i}} d x .
$$

Since $\sum_{i=1}^{N} \frac{\partial v_{i}}{\partial x_{i}}=\operatorname{div} \mathbf{v}=0$, we finally conclude that

$$
\begin{aligned}
\mathbf{v}(y)= & \int_{\Omega} \nabla H(y-x) \wedge \boldsymbol{\omega}(x) d x \\
& +\int_{\partial \Omega} \mathbf{v}(x) \nabla H(y-x) \cdot \mathbf{n} d \sigma-\int_{\partial \Omega} H(y-x) \nabla \sum_{i=1}^{N}\left(v_{i} n_{i}\right)(x) d \sigma .
\end{aligned}
$$

The last two integrals represent a harmonic vector $\mathbf{A}(y)$ in $\Omega$, since $y \neq x \in$ $\partial \Omega$ in the classical sense.
In the following, mainly for the sake of notational simplicity, we make use of tensors. For an introduction to these objects, see for example [1].
Definition 17.2. Let $\mathbf{k}$ be a $N$-vector defined in $\mathbb{R}^{N} \times \mathbb{R}$ and $\mathbf{g}$ an $M$-tensor defined in $\Omega_{t_{1}, t_{2}}$. Then the convolution $\mathbf{k} * \mathbf{g}$ is a $(M-1)$-tensor defined in $\Omega_{t_{1}, t_{2}}$ with components

$$
(\mathbf{k} * \mathbf{g})_{l m}=\iint_{\Omega_{t_{1}, t_{2}}} \sum_{i=1}^{N} k_{i}(x-\xi, t-\tau) g_{i l m}(\xi, \tau) d \xi d \tau
$$

Moreover, we let

$$
\operatorname{div} \mathbf{g} \stackrel{\text { def }}{=} \sum_{i=1}^{N} \frac{\partial g_{i l m}}{\partial x_{i}} .
$$

We have the following
Proposition 17.1 Assume that $\mathbf{v}$ is a local, weak solution of the NavierStokes equations in $E_{T}, \mathbf{v} \in L^{2,2}\left(E_{T} ; \mathbb{R}^{N}\right)$, $\boldsymbol{\omega} \in L^{2,2}\left(E_{T}\right)$, and that $\mathbf{f}$ is conservative with $\mathbf{f} \in L^{1,1}\left(E_{T} ; \mathbb{R}^{N}\right)$. Then in any $\Omega_{t_{1}, t_{2}} \subset \subset E_{T}$ we have

$$
\begin{equation*}
\boldsymbol{\omega}=\nabla \Gamma * \mathbf{g}+\mathbf{B} \tag{17.2}
\end{equation*}
$$

where $\Gamma$ is the function of (16.3), $\mathbf{g}=(N-1) \boldsymbol{\omega} \wedge \mathbf{v}$, and $\mathbf{B}=\mathbf{B}(x, t)$ is a solution of the heat equation (16.2) in $\Omega_{t_{1}, t_{2}}$.
Proof. We initially assume that $\mathbf{v}$ and $\boldsymbol{\omega}$ are both of class $C^{2}$, in order to easily perform some of the computations to follow.

First of all, it is a matter of straightforward calculations, to check that in our case

$$
(\mathbf{v} \cdot \nabla) \mathbf{v}=\operatorname{div}(\mathbf{v} \otimes \mathbf{v})
$$

If we now denote by $\mathbf{v}_{h}=\mathbf{v}_{h}(x, t)$ an integral average of $\mathbf{v}$ over a ball in space-time of radius $h$ centered at $(x, t)$, it follows from (8.2) that there must exist a regular function $p_{h}$, such that

$$
\begin{equation*}
\partial_{t} \mathbf{v}_{h}-\nu \Delta \mathbf{v}_{h}=-\operatorname{div}(\mathbf{v} \otimes \mathbf{v})_{h}+\mathbf{f}_{h}-\nabla p_{h} \tag{17.3}
\end{equation*}
$$

If we take the curl of all the terms in the previous equation, and switch the derivation order, we obtain

$$
\partial_{t} \boldsymbol{\omega}_{h}-\nu \Delta \boldsymbol{\omega}_{h}=-\operatorname{curl} \operatorname{div}(\mathbf{v} \otimes \mathbf{v})_{h},
$$

where we have taken into account that curl $\mathbf{f}_{h}=0$ since $\mathbf{f}$ is conservative. Again, it is a matter of straightforward computations to see that
$-\operatorname{curl} \operatorname{div}(\mathbf{v} \otimes \mathbf{v})_{h}=\operatorname{div}((N-1) \operatorname{curl} \mathbf{v} \wedge \mathbf{v})_{h}=\operatorname{div}((N-1) \boldsymbol{\omega} \wedge \mathbf{v})_{h}=\operatorname{div} \mathbf{g}_{h}$, so that we can write

$$
\begin{equation*}
\partial_{t} \boldsymbol{\omega}_{h}-\nu \Delta \boldsymbol{\omega}_{h}=\operatorname{div} \mathbf{g}_{h} \tag{17.4}
\end{equation*}
$$

Now, let

$$
\mathbf{B}_{h} \stackrel{\text { def }}{=} \boldsymbol{\omega}_{h}-\nabla \Gamma * \mathbf{g}_{h} ;
$$

we have

$$
\begin{aligned}
\partial_{t} \mathbf{B}_{h} & =\partial_{t} \boldsymbol{\omega}_{h}-\partial_{t}\left(\nabla \Gamma * \mathbf{g}_{h}\right)=\partial_{t} \boldsymbol{\omega}_{h}-\left(\nabla \partial_{t} \Gamma\right) * \mathbf{g}_{h} \\
\nu \Delta \mathbf{B}_{h} & =\nu \Delta \boldsymbol{\omega}_{h}-\nu \Delta\left(\nabla \Gamma * \mathbf{g}_{h}\right)=\nu \Delta \boldsymbol{\omega}_{h}-\nu(\nabla \Delta \Gamma) * \mathbf{g}_{h}
\end{aligned}
$$

Hence, since $\Gamma$ is the fundamental equation of (16.2), due to (17.4) we conclude that

$$
\begin{aligned}
\partial_{t} \mathbf{B}_{h}-\nu \Delta \mathbf{B}_{h} & =\partial_{t} \boldsymbol{\omega}_{h}-\nu \Delta \boldsymbol{\omega}_{h}-\nabla\left(\partial_{t} \Gamma-\nu \Delta \Gamma\right) * \mathbf{g}_{h} \\
& =\partial_{t} \boldsymbol{\omega}_{h}-\nu \Delta \boldsymbol{\omega}_{h}-\left(\partial_{t} \Gamma-\nu \Delta \Gamma\right) * \operatorname{div} \mathbf{g}_{h} \\
& =\partial_{t} \boldsymbol{\omega}_{h}-\nu \Delta \boldsymbol{\omega}_{h}-\operatorname{div} \mathbf{g}_{h} \\
& =0
\end{aligned}
$$

As $\mathbf{B}_{h}$ is in $L^{1,1}\left(E_{T} ; \mathbb{R}^{N}\right)$ uniformly with respect to $h$, we can then pass to the limit as $h \rightarrow 0$ and conclude. If $\mathbf{v}$ and $\boldsymbol{\omega}$ are not in $C^{2}$, the previous computations can be concluded by standard limiting arguments.

## 18 Proof of Theorem 15.1 Concluded

As a consequence of Proposition 17.1, in any $\Omega_{t_{1}, t_{2}} \subset \subset E_{T}$, we can write

$$
\boldsymbol{\omega}=\nabla \Gamma * \mathbf{g}+\mathbf{B}
$$

where $\mathbf{g}=(N-1) \boldsymbol{\omega} \wedge \mathbf{v}$. Hence, $|\mathbf{g}| \leq \gamma|\boldsymbol{\omega}||\mathbf{v}|$, where $\gamma$ depends only on $N$.
We first prove that in any $\Omega_{t_{1}, t_{2}} \subset \subset E_{T}$ we have $\boldsymbol{\omega} \in L^{\infty}\left(\Omega_{t_{1}, t_{2}}\right)$. If $\mathbf{v} \in$ $L^{p, q}\left(E_{T} ; \mathbb{R}^{N}\right)$ and $\boldsymbol{\omega} \in L^{r, s}\left(E_{T}\right)$, then, by Hölder's inequality, $\mathbf{g} \in L^{\rho, \sigma}\left(E_{T}\right)$, where $\rho, \sigma \geq 1$ are given by

$$
\frac{1}{\rho}=\frac{1}{p}+\frac{1}{r}, \quad \frac{1}{\sigma}=\frac{1}{q}+\frac{1}{s} .
$$

We define the positive constant $\kappa \in(0,1)$ by

$$
(N+3) \kappa=1-\left(\frac{N}{p}+\frac{2}{q}\right)
$$

and also

$$
\varrho=\frac{r}{1-\kappa r}, \quad \varsigma=\frac{s}{1-\kappa s},
$$

where $\varrho=\infty$ if $\kappa r \geq 1$, and analogously $\varsigma=\infty$ if $\kappa s \geq 1$. It is straightforward to check that

$$
1 \leq \rho \leq \varrho, \quad 1 \leq \sigma \leq \varsigma, \quad N\left(\frac{1}{\rho}-\frac{1}{\varrho}\right)+2\left(\frac{1}{\sigma}-\frac{1}{\varsigma}\right)=1-\kappa<1
$$

By Lemma 16.1 and Proposition 17.1, we conclude that $\boldsymbol{\omega} \in L^{\varrho, \varsigma}\left(E_{T}\right)$, where $\varrho, \varsigma$ are larger than $r$, $s$, so that $\boldsymbol{\omega}$ actually enjoys a higher integrability with respect to what originally assumed. The process can be repeated an arbitrary number of times, beginning with $r=s=2$. After a finite number of steps, one has $\boldsymbol{\omega} \in L^{\varrho, \varsigma}\left(E_{T}\right)$ with $\varrho=\varsigma \geq \kappa^{-1}$; at the next step $\varrho=\varsigma=\infty$, and we have finished the first part of the proof.

By Theorem 17.1, we now have

$$
\begin{equation*}
\mathbf{v}(y, t)=\int_{\Omega} \nabla_{x} H(y-x) \wedge \boldsymbol{\omega}(x, t) d x+\boldsymbol{A}(y, t) \tag{18.1}
\end{equation*}
$$

where $\mathbf{v} \in L^{2, \infty}\left(\Omega_{t_{1}, t_{2}} ; \mathbb{R}^{N}\right)$, and we have just proven that $\boldsymbol{\omega} \in L^{\infty}\left(\Omega_{t_{1}, t_{2}}\right)$. Hence, the function $\boldsymbol{A}=\boldsymbol{A}(x, t)$ must be bounded on compact subsets of $\Omega$, both as a function of $x$ and of $t$, and consequently, $\mathbf{v} \in L^{\infty}\left(\Omega_{t_{1}, t_{2}} ; \mathbb{R}^{N}\right)$. By the usual potential theoretic estimates for heat kernel convolutions (see for example [53]), $\boldsymbol{\omega}$ is Hölder continuous with respect to the space variables in any compact subregion of $E_{T}$, with arbitrary exponent $\alpha \in(0,1)$. By the Hölder continuity of $\boldsymbol{\omega}$ and (18.1), we have that also $\mathbf{v}$ is Hölder continuous.

This yields that $\mathbf{g}$ is Hölder continuous, and by the same potential theoretic estimates for the heat kernel convolution we have just relied upon, we have
that $\nabla \boldsymbol{\omega}$ is Hölder continuous. From here on, we can bootstrap, and conclude that $\mathbf{v} \in C^{\infty}$ with respect to the space variables.

Up to now, we have not used yet that $\mathbf{v}_{t} \in L^{2, s}$ with $s \geq 1$. It is rather straightforward to show that (17.2) implies

$$
\begin{equation*}
\partial_{t} \boldsymbol{\omega}-\nu \Delta \boldsymbol{\omega}=\operatorname{div} \mathbf{g} \tag{18.2}
\end{equation*}
$$

In turn this yields that $\partial_{t} \boldsymbol{\omega}$ is of class $C^{\infty}$ in the space variables, and its derivatives are bounded on compact subsets of $E_{T}$. On the other hand, if we differentiate (18.1) with respect to time, we have

$$
\mathbf{v}_{t}(y, t)=\int_{\Omega} \nabla_{x} H(y-x) \wedge \boldsymbol{\omega}_{t}(x, t) d x+\boldsymbol{A}_{t}(y, t)
$$

Thus, $\mathbf{v}_{t}$ is of class $C^{\infty}$ in the space variables, and each derivative is of class $L^{s}$ in time. Finally, we recover that equation (15.1) holds almost everywhere in $E_{T}$, by letting $h \rightarrow 0$ in (17.3).

## 19 Regularity of the Initial-Boundary Value Problem

As we have already discussed in Section 15, solutions of the Navier-Stokes equations behave globally with respect to time, as they are instantaneously determined by the boundary conditions, but they are somehow purely local as far as the space variables are concerned. This suggests that one can hope to gain time regularity from the assigned initial-boundary value problem. We will not go into details here, and we limit to state a result, whose proof can be found in $[17, \S 5]$.

Theorem 19.1. Let $\mathbf{v}$ be a weak solution in $E_{T}$ of the initial-boundary value problem (8.1) with $\mathbf{f} \equiv 0$ and $\mathbf{v}_{o} \in H$. Assume that $\mathbf{v}$ satisfies at least one of the following two conditions:
(i) $\mathbf{v} \in L^{p, q}\left(E_{T} ; \mathbb{R}^{N}\right)$, for some $p, q$ such that $\frac{N}{p}+\frac{2}{q}=1, \quad p \in(N, \infty]$;
(ii) $\mathbf{v} \in C^{0}\left([0, T] ; L^{N}(E)\right)$.

Then, if $E$ is uniformly of class $C^{\infty}$, we have $\mathbf{v} \in C^{\infty}(\bar{E} \times(0, T])$.
Remark 19.1 For $E=\mathbb{R}^{3}$, Theorem 19.1 was first proved by Leray [27, pp. 224-227], while for $E=\mathbb{R}^{N}$ with $N \geq 2$, and $p<\infty$ it is due to [12]. Sohr proved Theorem 19.1(i) with $p<\infty$, for domains with a bounded boundary in [43]. That condition (ii) implies regularity was first discovered by von Wahl ([52]), in the case of a bounded domain. This latter result was extended to domains with a bounded boundary by Giga ([18]).

## 20 Recovering the Pressure in the Time-Dependent Equations

In Section 12 we have shown how a moderate degree of integrability of $\nabla p$ yields a higher regularity on $\mathbf{v}$. We return to this issue and discuss the regularity of $p$ when considering weak solutions of (8.1) for $N=3$. Instead of dealing with a general domain $E_{T}$, just for simplicity we work with $B_{1} \times(-1,0)$. Moreover, we take $\nu=1$. We will prove the following.
Proposition 20.1 (Sohr-von Wahl [45]) Let $\mathbf{v}_{o} \in L^{2}\left(B_{1} ; \mathbb{R}^{3}\right)$ be weakly divergence free in $B_{1}$ and $\mathbf{f} \equiv 0$. If $\mathbf{v}$ is the corresponding weak solution of (8.1) in $B_{1} \times(-1,0)$, then $p \in L^{\frac{5}{3}}\left(-1,0 ; L^{\frac{5}{3}}\left(B_{1}\right)\right)$.

Proof. If we rely on (12.1) written over $B_{1}$ with $p=\frac{15}{14}$ and $p q=2$, we have

$$
\begin{align*}
\|(\mathbf{v} \cdot \nabla) \mathbf{v}\|_{\frac{15}{14} ; B_{1}}^{\frac{5}{3}} & \leq\left(\int_{B_{1}}|\nabla \mathbf{v}|^{2} d x\right)^{\frac{5}{6}}\left(\int_{B_{1}}|\mathbf{v}|^{\frac{30}{13}}\right)^{\frac{13}{18}}  \tag{20.1}\\
& \leq C\left[\|\nabla \mathbf{v}\|_{2 ; B_{1}}^{2}+\|\mathbf{v}\|_{\frac{30}{13} ; B_{1}}^{10}\right]
\end{align*}
$$

Now we rely on the following interpolation inequality, which can be proven, for example, relying on Proposition 18.1 and Theorem 19.1 of Chapter IX of [6].
Lemma 20.1 Let $r>0$. For $v \in W^{1,2}\left(B_{r}\right)$ we have

$$
\begin{aligned}
\int_{B_{r}}|\mathbf{v}|^{q} d x \leq & C\left[\int_{B_{r}}|\nabla \mathbf{v}|^{2} d x\right]^{a}\left[\int_{B_{r}}|\mathbf{v}|^{2} d x\right]^{\frac{q}{2}-a} \\
& +\frac{C}{r^{2 a}}\left[\int_{B_{r}}|\mathbf{v}|^{2} d x\right]^{\frac{q}{2}}
\end{aligned}
$$

for all $q \in[2,6], a=\frac{3(q-2)}{4}$.
If we choose $q=\frac{30}{13}$ and $a=\frac{3}{13}$ in Lemma 20.1, we obtain

$$
\|\mathbf{v}\|_{\frac{30}{13} ; B_{1}} \leq C\|\nabla \mathbf{v}\|_{2 ; B_{1}}^{\frac{1}{5}}\|\mathbf{v}\|_{2 ; B_{1}}^{\frac{4}{5}}+C\|\mathbf{v}\|_{2 ; B_{1}}
$$

and also,

$$
\begin{equation*}
\|\mathbf{v}\|_{\frac{30}{13} ; B_{1}}^{10} \leq C\|\nabla \mathbf{v}\|_{2 ; B_{1}}^{2}\|\mathbf{v}\|_{2 ; B_{1}}^{8}+C\|\mathbf{v}\|_{2 ; B_{1}}^{10} \tag{20.2}
\end{equation*}
$$

If we take both (20.1) and (20.2) into account, we conclude that

$$
\|(\mathbf{v} \cdot \nabla) \mathbf{v}\|_{\frac{15}{14} ; B_{1}}^{\frac{5}{3}} \leq C\left[\|\nabla \mathbf{v}\|_{2 ; B_{1}}^{2}+\|\nabla \mathbf{v}\|_{2 ; B_{1}}^{2}\|\mathbf{v}\|_{2 ; B_{1}}^{8}+\|\mathbf{v}\|_{2 ; B_{1}}^{10}\right]
$$

and integrating with respect to time over $(-1,0)$ yields

$$
\begin{aligned}
& \int_{-1}^{0}\left(\int_{B_{1}}|(\mathbf{v} \cdot \nabla) \mathbf{v}|^{\frac{15}{14}} d x\right)^{\frac{14}{15} \cdot \frac{5}{3}} d t \\
& \leq C \int_{-1}^{0} \int_{B_{1}}|\nabla \mathbf{v}|^{2} d x d t \\
& \quad+C \int_{-1}^{0}\left(\int_{B_{1}}|\nabla \mathbf{v}|^{2} d x\right)\left(\int_{B_{1}}|\mathbf{v}|^{2} d x\right)^{4} d t \\
& \quad+C \int_{-1}^{0}\left(\int_{B_{1}}|\mathbf{v}|^{2} d x\right)^{5} d t,
\end{aligned}
$$

where all the terms on the right-hand side are bounded, since $\mathbf{v} \in W$. Hence, we conclude that

$$
\begin{equation*}
(\mathbf{v} \cdot \nabla) \mathbf{v} \in L^{\frac{5}{3}}\left(-1,0 ; L^{\frac{15}{14}}\left(B_{1}\right)\right) \tag{20.3}
\end{equation*}
$$

Take $\varphi \in \mathcal{V}$, where in (8.2) we assume $E=B_{1}$. If we use such a $\varphi$ in the weak formulation of Navier-Stokes equations, we obtain

$$
\begin{aligned}
\left(\frac{\partial \mathbf{v}}{\partial t}, \boldsymbol{\varphi}\right) & =-(\nabla \mathbf{v}, \nabla \boldsymbol{\varphi})-((\mathbf{v} \cdot \nabla) \mathbf{v}, \boldsymbol{\varphi}) \\
\left|\left(\frac{\partial \mathbf{v}}{\partial t}, \boldsymbol{\varphi}\right)\right| & =|-(\nabla \mathbf{v}, \nabla \boldsymbol{\varphi})-((\mathbf{v} \cdot \nabla) \mathbf{v}, \boldsymbol{\varphi})| \\
& \leq\|\nabla \mathbf{v}(\cdot, t)\|_{2 ; B_{1}}\|\nabla \boldsymbol{\varphi}\|_{2 ; B_{1}}+\|\mathbf{v}(\cdot, t)\|_{2 ; B_{1}}\|\nabla \mathbf{v}(\cdot, t)\|_{2 ; B_{1}}\|\boldsymbol{\varphi}\|_{2 ; B_{1}} \\
& \leq\left[\|\nabla \mathbf{v}(\cdot, t)\|_{2 ; B_{1}}+\|\mathbf{v}(\cdot, t)\|_{2 ; B_{1}}\|\nabla \mathbf{v}(\cdot, t)\|_{2 ; B_{1}}\right]\|\boldsymbol{\varphi}\|_{W^{2,2}\left(B_{1}\right)}
\end{aligned}
$$

Hence,

$$
\frac{\partial \mathbf{v}}{\partial t}-\Delta \mathbf{v} \in L^{2}(-1,0 ; Z)
$$

where $Z$ is the dual space of $W_{0}^{2,2}\left(B_{1}\right)$. We define

$$
\mathbf{g}=\frac{\partial \mathbf{v}}{\partial t}-\Delta \mathbf{v}
$$

and notice that for almost every $t \in(-1,0)$,

$$
\operatorname{div} \mathbf{g}=\frac{\partial}{\partial t}(\operatorname{div} \mathbf{v})-\Delta(\operatorname{div} \mathbf{v})=0, \quad \operatorname{curl} \mathbf{g}=\operatorname{curl}((\mathbf{v} \cdot \nabla) \mathbf{v}) \quad \text { in } B_{1}
$$

Then, by the elliptic estimates of [32], Chapter 7,

$$
\|\mathbf{g}\|_{\frac{15}{14}, B_{1}}^{\frac{5}{3}} \leq C\left[\|(\mathbf{v} \cdot \nabla) \mathbf{v}\|_{\frac{15}{14}, B_{1}}^{\frac{5}{3}}+\|\mathbf{g}\|_{Z}^{\frac{5}{3}}\right]
$$

Therefore, integrating we have

$$
\begin{equation*}
\frac{\partial \mathbf{v}}{\partial t}-\Delta \mathbf{v} \in L^{\frac{5}{3}}\left(-1,0 ; L^{\frac{15}{14}}\left(B_{1}\right)\right) \tag{20.4}
\end{equation*}
$$

(20.3)-(20.4) imply that

$$
\nabla p \in L^{\frac{5}{3}}\left(-1,0 ; L^{\frac{15}{14}}\left(B_{1}\right)\right)
$$

and by the Sobolev embedding theorem, we conclude that

$$
p \in L^{\frac{5}{3}}\left(-1,0 ; L^{\frac{5}{3}}\left(B_{1}\right)\right)
$$

Remark 20.1 The proof of Proposition 20.1 is taken from [29].

## 21 How the Quantities Scale in the Equations

Concerning the quantities in the Navier-Stokes equations, if we rewrite

$$
\mathbf{v}_{t}-\nu \Delta \mathbf{v}+(\mathbf{v} \cdot \nabla) \mathbf{v}+\nabla p=\mathbf{f}
$$

taking into account only the physical dimensions, we have

$$
\frac{[v]}{[T]}-\frac{[v]}{[L]^{2}}+\frac{[v]^{2}}{[L]}+\frac{[p]}{[L]}=[f] .
$$

This easily yields

$$
\begin{aligned}
& \operatorname{dim}[L]=1 \\
& \operatorname{dim}[T]=2 \\
& \frac{[v]}{[L]}=[v]^{2} \quad \Rightarrow \quad \operatorname{dim}[v]=-1 \\
& {[p]=[v]^{2} \quad \Rightarrow \quad \operatorname{dim}[p]=-2} \\
& {[f]=\frac{[p]}{[L]} \quad \Rightarrow \quad \operatorname{dim}[f]=-3}
\end{aligned}
$$

This will be very useful in the next Sections.

## 22 The Generalized or Localized Energy Inequality

In the following we work with homogeneous Navier-Stokes equations, that is, we take $\mathbf{f}=0$. Moreover, for the sake of simplicity, we assume $\nu=1$ (there is no loss of generality in this assumption, as we have pointed out before more than once).

We have already discussed the notion of weak solution in the sense of Leray-Hopf. At this stage, it is perhaps useful to recall how the initial condition $\mathbf{v}_{o} \in L^{2}(E)$ is assumed: we have

$$
\lim _{t \rightarrow 0}\left\|\mathbf{v}(\cdot, t)-\mathbf{v}_{o}\right\|_{L^{2}(E)}=0
$$

Since in the following we want to develop a local regularity theory, instead of the global energy inequality typical of Leray-Hopf's solutions, we need a localized version.

As typical when introducing weak notions of solutions, we first assume $(\mathbf{v}, p)$ to be regular, we deduce the corresponding inequality, and then we take it as a definition.

Consider a non-negative function $\varphi \in C_{o}^{\infty}(E \times(0, T) ; \mathbb{R})$, multiply the equation by $\varphi \mathbf{v}$, and integrate. We have

$$
\begin{aligned}
& \int_{0}^{T} \int_{E} \mathbf{v}_{t} \cdot(\varphi \mathbf{v}) d x d t+\int_{0}^{T} \int_{E} \nabla \mathbf{v}: \nabla(\varphi \mathbf{v}) d x d t \\
& +\int_{0}^{T} \int_{E}(\varphi \mathbf{v}) \cdot(\mathbf{v} \cdot \nabla) \mathbf{v} d x d t+\int_{0}^{t} \int_{E} \nabla p \cdot(\varphi \mathbf{v}) d x d t \\
& =I_{1}+I_{2}+I_{3}+I_{4}=0
\end{aligned}
$$

Notice that we are not requiring that $\operatorname{div} \varphi=0$. We consider all the terms one by one.

$$
\begin{aligned}
I_{1}= & \int_{0}^{T} \int_{E} \mathbf{v}_{t} \cdot(\varphi \mathbf{v}) d x d t=\frac{1}{2} \int_{0}^{T} \int_{E} \varphi \partial_{t}|\mathbf{v}|^{2} d x d t \\
= & \frac{1}{2} \int_{E \times\{T\}}|v|^{2} \varphi d x-\frac{1}{2} \int_{E \times\{0\}}|\mathbf{v}|^{2} \varphi d x \\
& -\frac{1}{2} \int_{0}^{T} \int_{E}|\mathbf{v}|^{2} \partial_{t} \varphi d x d t .
\end{aligned}
$$

The first two terms cancel because of the definition of $\varphi$. Since

$$
\nabla(\varphi \mathbf{v})=\varphi \nabla \mathbf{v}+\mathbf{v} \nabla \varphi
$$

we have

$$
\begin{aligned}
I_{2}= & \int_{0}^{T} \int_{E} \nabla \mathbf{v}: \nabla(\varphi \mathbf{v}) d x d t=\int_{0}^{T} \int_{E} \varphi|\nabla \mathbf{v}|^{2} d x d t \\
& +\int_{0}^{T} \int_{E} \nabla \mathbf{v}:(\mathbf{v} \nabla \varphi) d x d t=\int_{0}^{T} \int_{E} \varphi|\nabla \mathbf{v}|^{2} d x d t \\
& +\frac{1}{2} \int_{0}^{T} \int_{E} \nabla|\mathbf{v}|^{2} \cdot \nabla \varphi d x d t \\
= & \int_{0}^{T} \int_{E} \varphi|\nabla \mathbf{v}|^{2} d x d t-\frac{1}{2} \int_{0}^{T} \int_{E}|\mathbf{v}|^{2} \Delta \varphi d x d t \\
I_{4}= & \int_{0}^{T} \int_{E} \nabla p \cdot(\varphi \mathbf{v}) d x d t=-\int_{0}^{T} \int_{E} p \operatorname{div}(\varphi \mathbf{v}) d x d t \\
= & -\int_{0}^{T} \int_{E}^{P} \operatorname{div} \mathbf{v} d x d t-\int_{0}^{T} \int_{E} p \mathbf{v} \cdot \nabla \varphi d x d t .
\end{aligned}
$$

Finally

$$
I_{3}=\int_{0}^{T} \int_{E}(\varphi \mathbf{v}) \cdot(\mathbf{v} \cdot \nabla) \mathbf{v} d x d t=-\frac{1}{2} \int_{0}^{T} \int_{E}|\mathbf{v}|^{2} \mathbf{v} \cdot \nabla \varphi d x d t
$$

The result is a consequence of the following fact. By sheer computations, we have

$$
\begin{aligned}
(\varphi \mathbf{v}) & \cdot(\mathbf{v} \cdot \nabla) \mathbf{v} \\
= & \left(\varphi v_{1}, \varphi v_{2}, \ldots, \varphi v_{N}\right) \cdot\left[v_{1} \partial_{x_{1}}+\cdots+v_{N} \partial_{x_{N}}\right]\left(v_{1}, \ldots, v_{N}\right) \\
= & \varphi v_{1}\left[v_{1} \partial_{x_{1}} v_{1}+v_{2} \partial_{x_{2}} v_{1}+\cdots+v_{N} \partial_{x_{N}} v_{1}\right] \\
& +\ldots \\
& +\varphi v_{N}\left[v_{1} \partial_{x_{1}} v_{N}+v_{2} \partial_{x_{2}} v_{N}+\cdots+v_{N} \partial_{x_{N}} v_{N}\right] \\
= & \varphi v_{1}\left[\partial_{x_{1}} \frac{1}{2} v_{1}^{2}+v_{2} \partial_{x_{2}} v_{1}+\cdots+v_{N} \partial_{x_{N}} v_{1}\right] \\
& +\varphi v_{2}\left[v_{1} \partial_{x_{1}} v_{2}+\partial_{x_{2}} \frac{1}{2} v_{2}^{2}+\cdots+v_{N} \partial_{x_{N}} v_{2}\right] \\
& +\ldots \\
& +\varphi v_{N}\left[v_{1} \partial_{x_{1}} v_{N}+v_{2} \partial_{x_{2}} v_{N}+\cdots+\frac{1}{2} \partial_{x_{N}} v_{N}^{2}\right] \\
= & \varphi \\
& {\left[v_{1} \partial_{x_{1}} \frac{1}{2} v_{1}^{2}+v_{2} \partial_{x_{2}} \frac{1}{2} v_{1}^{2}+\cdots+v_{N} \partial_{x_{N}} \frac{1}{2} v_{1}^{2}\right] } \\
& +\varphi\left[v_{1} \partial_{x_{1}} \frac{1}{2} v_{2}^{2}+v_{2} \partial_{x_{2}} \frac{1}{2} v_{2}^{2}+\cdots+v_{2} \partial_{x_{N}} \frac{1}{2} v_{N}^{2}\right] \\
& +\ldots \\
& +\varphi\left[v_{1} \partial_{x_{1}} \frac{1}{2} v_{N}^{2}+v_{2} \partial_{x_{2}} \frac{1}{2} v_{N}^{2}+\cdots+v_{N} \partial_{x_{N}} \frac{1}{2} v_{N}^{2}\right] \\
= & \varphi v_{1} \partial_{x_{1}} \frac{1}{2}|\mathbf{v}|^{2}+\varphi v_{2} \partial_{x_{2}} \frac{1}{2}|\mathbf{v}|^{2}+\cdots+\varphi v_{N} \partial_{x_{N}} \frac{1}{2}|\mathbf{v}|^{2}=\varphi \mathbf{v} \cdot \nabla \frac{1}{2}|\mathbf{v}|^{2} .
\end{aligned}
$$

Hence, if we take into account that we are integrating with respect to space and time, relying on the previous computations we have

$$
\begin{aligned}
\int_{0}^{T} & \int_{E}(\varphi \mathbf{v}) \cdot(\mathbf{v} \cdot \nabla) \mathbf{v} d x d t \\
& =\int_{0}^{T} \int_{E} \varphi \mathbf{v} \cdot \nabla \frac{1}{2}|\mathbf{v}|^{2} d x d t=-\int_{0}^{T} \int_{E} \frac{1}{2}|\mathbf{v}|^{2} \operatorname{div}(\varphi \mathbf{v}) d x d t \\
& =-\left.\int_{0}^{T} \int_{E} \frac{1}{2}\left|\mathbf{v}^{2} \varphi \operatorname{div} \mathbf{v} d x d t-\int_{0}^{T} \int_{E} \frac{1}{2}\right| \mathbf{v}\right|^{2} \mathbf{v} \cdot \nabla \varphi d x d t
\end{aligned}
$$

Eventually, if we collect all the terms, we have

$$
\begin{aligned}
& \int_{0}^{T} \int_{E}|\nabla \mathbf{v}|^{2} \varphi d x d t=\frac{1}{2} \int_{0}^{T} \int_{E}|\mathbf{v}|^{2}\left(\partial_{t} \varphi+\Delta \varphi\right) d x d t \\
& +\frac{1}{2} \int_{0}^{T} \int_{E}|\mathbf{v}|^{2} \mathbf{v} \cdot \nabla \varphi d x d t+\int_{0}^{T} \int_{E} p \mathbf{v} \cdot \nabla \varphi d x d t
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
& 2 \int_{0}^{T} \int_{E}|\nabla \mathbf{v}|^{2} \varphi d x d t=\int_{0}^{T} \int_{E}|\mathbf{v}|^{2}\left(\partial_{t} \varphi+\Delta \varphi\right) d x d t \\
& +\int_{0}^{T} \int_{E}\left(|\mathbf{v}|^{2}+2 p\right) \mathbf{v} \cdot \nabla \varphi d x d t
\end{aligned}
$$

We say that a weak solution ( $\mathbf{v}, p$ ) satisfies the generalized (or localized) energy inequality if

$$
\begin{aligned}
2 \int_{0}^{T} \int_{E}|\nabla \mathbf{v}|^{2} \varphi d x d t \leq & \int_{0}^{T} \int_{E}|\mathbf{v}|^{2}\left(\partial_{t} \varphi+\Delta \varphi\right) d x d t \\
& +\int_{0}^{T} \int_{E}\left(|\mathbf{v}|^{2}+2 p\right) \mathbf{v} \cdot \nabla \varphi d x d t
\end{aligned}
$$

for all non-negative $\varphi \in C_{o}^{\infty}(E \times(0, T) ; \mathbb{R})$. We will say something about the summability of $\mathbf{v}$ in a while. We mentioned before the weak continuity in $L^{2}(E)$ of $\mathbf{v}(\cdot, t)$; this means that $\forall \psi \in L^{2}(E)$, we have

$$
\int_{E} \mathbf{v}(\cdot, t) \psi d x \quad \rightarrow \quad \int_{E} \mathbf{v}\left(\cdot, t_{o}\right) \psi d x
$$

as $t \rightarrow t_{o} \in[0, T]$. As a consequence of this, the generalized energy inequality can be further localized with respect to time; indeed, $\forall t \in(0, T), \forall \varphi \in$ $C_{0}^{\infty}(E \times(0, T) ; \mathbb{R}), \varphi \geq 0$

$$
\begin{align*}
& \int_{E \times\{t\}}|\mathbf{v}|^{2} \varphi d x+2 \int_{0}^{t} \int_{E}|\nabla \mathbf{v}|^{2} \varphi d x d \tau \\
& \leq \int_{0}^{t} \int_{E}|\mathbf{v}|^{2}\left(\partial_{\tau} \varphi+\Delta \varphi\right) d x d \tau+\int_{0}^{t} \int_{E}\left(|\mathbf{v}|^{2}+2 p\right) \mathbf{v} \cdot \nabla \varphi d x d \tau \tag{22.1}
\end{align*}
$$

## 23 An Introductory Estimate

In the sequel we will need the following introductory result.
Lemma 23.1 Let $\mathbf{v} \in L^{\infty}\left(0, T ; L^{2}(E)\right) \cap L^{2}\left(0, T ; W^{1,2}(E)\right),\left(x_{o}, t_{o}\right) \in E_{T}$, assume that $B_{\rho}\left(x_{o}\right) \times\left(t_{o}-\rho^{2}, t_{o}\right] \subset E_{T}$, and for $0<r \leq \rho$ let

$$
A(r) \stackrel{\text { def }}{=} \sup _{t_{o}-r^{2}<t<t_{o}} \frac{1}{r} \int_{B_{r}\left(x_{o}\right)}|\mathbf{v}(\cdot, t)|^{2} d x
$$

$$
\begin{aligned}
& B(r) \stackrel{\text { def }}{=} \frac{1}{r} \int_{t_{o}-r^{2}}^{t_{o}} \int_{B_{\rho}\left(x_{o}\right)}|\nabla \mathbf{v}|^{2} d x d t \\
& C(r) \stackrel{\text { def }}{=} \frac{1}{r^{2}} \int_{t_{o}-r^{2}}^{t_{o}} \int_{B_{\rho}\left(x_{o}\right)}|\mathbf{v}|^{3} d x d t
\end{aligned}
$$

Then

$$
\begin{equation*}
C(r) \leq \gamma\left[\left(\frac{r}{\rho}\right)^{3}[A(\rho)]^{\frac{3}{2}}+\left(\frac{\rho}{r}\right)^{3}[A(\rho)]^{\frac{3}{4}}[B(\rho)]^{\frac{3}{4}}\right] \tag{23.1}
\end{equation*}
$$

where $\gamma$ depends only on the dimension $N=3$.
Proof. Without loss of generality, we may assume $\left(x_{o}, t_{o}\right)=(0,0)$. We let

$$
B_{r}=\{|x|<r\}, \quad Q_{r}=B_{r} \times\left(-r^{2}, 0\right]
$$

and

$$
\left|\overline{\mathbf{v}}_{\rho}\right|^{2}=\int_{B_{\rho}}|\mathbf{v}|^{2} d x
$$

for a.e $t \in\left(-r^{2}, 0\right]$ we have

$$
\begin{aligned}
\int_{B_{r}}|\mathbf{v}(\cdot, t)|^{2} d x & =\int_{B_{r}}\left[|\mathbf{v}|^{2}-\left|\overline{\mathbf{v}}_{\rho}\right|^{2}+\left|\overline{\mathbf{v}}_{\rho}\right|^{2}\right] d x \\
& \leq \int_{B_{\rho}} \|\left.\mathbf{v}\right|^{2}-\left.\left|\overline{\mathbf{v}}_{\rho}\right|^{2}\left|d x+\int_{B_{r}}\right| \overline{\mathbf{v}}_{\rho}\right|^{2} d x \\
& \leq\left. c_{1} \rho \int_{B_{\rho}}|\nabla| \mathbf{v}\right|^{2} \left\lvert\, d x+\frac{c_{2}}{\rho^{N}}\left(\int_{B_{\rho}}|\mathbf{v}|^{2} d x\right) r^{N}\right. \\
& \leq c_{3} \rho \int_{B_{\rho}}|\mathbf{v}||\nabla \mathbf{v}| d x+c_{2} \frac{r^{N}}{\rho^{N}}\left(\int_{B_{\rho}}|\mathbf{v}|^{2} d x\right) \\
& =c_{3} \rho \int_{B_{\rho}}|\mathbf{v}||\nabla \mathbf{v}| d x+c_{2}\left(\frac{r}{\rho}\right)^{N} \int_{B_{\rho}}|\mathbf{v}|^{2} d x
\end{aligned}
$$

Thus, we can conclude that

$$
\begin{align*}
\int_{B_{r}}|\mathbf{v}|^{2} d x \leq & c_{3} \rho^{\frac{3}{2}}\left[\frac{1}{\rho} \int_{B_{\rho}}|\mathbf{v}|^{2} d x\right]^{\frac{1}{2}}\left[\int_{B_{\rho}}|\nabla \mathbf{v}|^{2} d x\right]^{\frac{1}{2}} \\
& +c_{2}\left(\frac{r}{\rho}\right)^{N} \int_{B_{\rho}}|\mathbf{v}|^{2} d x  \tag{23.2}\\
\leq & c_{3} \rho^{\frac{3}{2}}[A(\rho)]^{\frac{1}{2}}\left[\int_{B_{\rho}}|\nabla \mathbf{v}|^{2} d x\right]^{\frac{1}{2}}+c_{2} \rho\left(\frac{r}{\rho}\right)^{N} A(\rho)
\end{align*}
$$

We now let $N=3$. If we take into account Lemma 20.1 and we choose $q=3$, $a=\frac{3}{4}$ we get

$$
\begin{equation*}
\int_{B_{r}}|\mathbf{v}|^{3} d x \leq \gamma\left[\int_{B_{r}}|\mathbf{v}|^{2} d x\right]^{\frac{3}{4}}\left[\int_{B_{r}}|\nabla \mathbf{v}|^{2} d x\right]^{\frac{3}{4}}+\frac{\gamma}{r^{\frac{3}{2}}}\left[\int_{B_{r}}|\mathbf{v}|^{2} d x\right]^{\frac{3}{2}} \tag{23.3}
\end{equation*}
$$

Combining (23.2) and (23.3) yields

$$
\begin{aligned}
\int_{B_{r}}|\mathbf{v}|^{3} d x \leq & c\left[\int_{B_{r}}|\mathbf{v}|^{2} d x\right]^{\frac{3}{4}}\left[\int_{B_{r}}|\nabla \mathbf{v}|^{2} d x\right]^{\frac{3}{4}} \\
& +\frac{c}{r^{\frac{3}{2}}}\left[\rho^{\frac{3}{2}}[A(\rho)]^{\frac{1}{2}}\left[\int_{B_{r}}|\nabla \mathbf{v}|^{2} d x\right]^{\frac{1}{2}}+\rho\left(\frac{r}{\rho}\right)^{3} A(\rho)\right]^{\frac{3}{2}} \\
\leq & c \rho^{\frac{3}{4}}[A(\rho)]^{\frac{3}{4}}\left[\int_{B_{\rho}}|\nabla \mathbf{v}|^{2} d x\right]^{\frac{3}{4}} \\
& +c \frac{\rho^{\frac{3}{4}}}{r^{\frac{3}{2}}}[A(\rho)]^{\frac{3}{4}}\left[\int_{B_{\rho}}|\nabla \mathbf{v}|^{2} d x\right]^{\frac{3}{4}}+\frac{c}{r^{\frac{3}{2}}} \rho^{\frac{3}{2}}\left(\frac{r}{\rho}\right)^{\frac{9}{2}}[A(\rho)]^{\frac{3}{2}} \\
= & c\left(\frac{r}{\rho}\right)^{3}[A(\rho)]^{\frac{3}{2}}+c\left[\rho^{\frac{3}{4}}+\frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}}\right][A(\rho)]^{\frac{3}{4}}\left[\int_{B_{\rho}}|\nabla \mathbf{v}|^{2} d x\right]^{\frac{3}{4}} .
\end{aligned}
$$

Now we integrate over $t$ in the interval $\left(-r^{2}, 0\right]$ to obtain

$$
\begin{aligned}
\int_{-r^{2}}^{0} \int_{B_{r}}|\mathbf{v}|^{3} d x d t \leq & c\left(\frac{r}{\rho}\right)^{3}[A(\rho)]^{\frac{3}{2}} r^{2} \\
& +c\left[\rho^{\frac{3}{4}}+\frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}}\right][A(\rho)]^{\frac{3}{4}} \int_{-\rho^{2}}^{0}\left[\int_{B_{\rho}}|\nabla \mathbf{v}|^{2} d x\right]^{\frac{3}{4}} d t \\
\leq & c r^{2}\left(\frac{r}{\rho}\right)^{3}[A(\rho)]^{\frac{3}{2}}+ \\
& +c\left(\rho^{\frac{3}{4}}+\frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}}\right)[A(\rho)]^{\frac{3}{4}} r^{\frac{1}{2}}\left[\int_{-\rho^{2}}^{0} \int_{B_{\rho}}|\nabla \mathbf{v}|^{2} d x d t\right]^{\frac{3}{4}}
\end{aligned}
$$

where we have applied the Hölder inequality (with respect to time) in order to estimate the last term. If we divide everything by $r^{2}$, we have

$$
\begin{aligned}
& \frac{1}{r^{2}} \int_{-r^{2}}^{0} \int_{B_{r}}|\mathbf{v}|^{3} d x d t \leq c\left(\frac{r}{\rho}\right)^{3}[A(\rho)]^{\frac{3}{2}} \\
&+\frac{c}{r^{\frac{3}{2}}}\left(\rho^{\frac{3}{4}}+\frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}}\right)[A(\rho)]^{\frac{3}{4}} \rho^{\frac{3}{4}}\left(\frac{1}{\rho} \int_{-\rho^{2}}^{0} \int_{B_{\rho}}|\nabla \mathbf{v}|^{2} d x d t\right)^{\frac{3}{4}}
\end{aligned}
$$

that is

$$
C(r) \leq c\left(\frac{r}{\rho}\right)^{3}[A(\rho)]^{\frac{3}{2}}+c\left(\frac{\rho^{\frac{3}{2}}}{r^{\frac{3}{2}}}+\frac{\rho^{3}}{r^{3}}\right)[A(\rho)]^{\frac{3}{4}}[B(\rho)]^{\frac{3}{4}}
$$

which finally yields

$$
C(r) \leq c\left(\frac{r}{\rho}\right)^{3}[A(\rho)]^{3 / 2}+c\left(\frac{\rho}{r}\right)^{3}[A(\rho)]^{3 / 4}[B(\rho)]^{3 / 4}
$$

Remark 23.1 Estimate (23.1) is simply a Real Analysis fact, and does not depend on $\mathbf{v}$ being a solution of the Navier-Stokes equations.

## 24 Suitable Weak Solutions and Partial Regularity

In their paper (see [3]) Caffarelli, Kohn \& Nirenberg introduce the notion of suitable weak solution, which we are going to define next.

Definition 24.1. A pair $(\mathbf{v}, p)$ is a suitable weak solution of the Navier-Stokes equations in an open set $D \subset \mathbb{R}^{3} \times \overline{\mathbb{R}}$ if the following conditions hold:

1. $(\mathbf{v}, p)$ satisfies the Navier-Stokes equations in the sense of distributions in $D$ [which is a much weaker assumption, with respect to what we usually require];
2. $p \in L^{\frac{3}{2}}(D)$ with $\iint_{D}|p|^{\frac{3}{2}} d x d t<E$ and for some constants $E_{o}$, $E_{1}$ we have

$$
\begin{aligned}
& \int_{D_{t}}|\mathbf{v}|^{2} d x \leq E_{o}, \quad D_{t}=D \cap\left(\mathbb{R}^{3} \times\{t\}\right) \quad \text { for a.e } t \text { such that } D_{t} \neq \emptyset \\
& \iint_{D}|\nabla \mathbf{v}|^{2} d x d t \leq E_{1}
\end{aligned}
$$

3. The generalized energy inequality (22.1) holds $\forall \varphi \in C_{o}^{\infty}\left(D ; \mathbb{R}_{+}\right)$.

Remark 24.1 With respect to the usual way of proceeding, now we have the extra condition about the pressure. A priori, it is hard to say whether solutions built by Leray and Hopf have the right summability as required here. However, as we have seen, $p \in L^{\frac{5}{3}}\left(Q_{1}\right)$ and this allows us to prove a suitable compactness result

Before coming to such a result, let us make few quick comments about the notion of suitable weak solution.

Remark 24.2 If we take the interpolation inequality of Lemma 20.1 in $B_{r} \times$ $(0, T]$ with $q=10 / 3$ (which yields $a=1$ ), we have

$$
\begin{aligned}
\int_{0}^{T} \int_{B_{r}}|\mathbf{v}|^{\frac{10}{3}} d x d t \leq & C \int_{0}^{T}\left[\int_{B_{r}}|\mathbf{v}|^{2} d x\right]^{\frac{2}{3}}\left[\int_{B_{r}}|\nabla \mathbf{v}|^{2} d x\right] d t \\
& +\frac{C}{r^{2}} \int_{0}^{T}\left(\int_{B_{r}}|\mathbf{v}|^{2} d x\right)^{\frac{5}{3}} d t \\
\leq & C\left(\sup _{0<t<T} \int_{B_{r}}|\mathbf{v}|^{2} d x\right)^{\frac{2}{3}} \int_{0}^{T} \int_{B_{r}}|\nabla \mathbf{v}|^{2} d x d t \\
& +\frac{C}{r^{2}}\left(\sup _{0<t<T} \int_{B_{r}}|\mathbf{v}|^{2} d x\right)^{\frac{5}{3}} \int_{0}^{T} d t \\
= & C E_{o}^{\frac{2}{3}} E_{1}+\frac{C}{r^{2}} E_{o}^{\frac{5}{3}} T
\end{aligned}
$$

In particular, this yields that $\int_{0}^{T} \int_{B_{r}}|\mathbf{v}|^{2} \mathbf{v} \cdot \nabla \varphi d x d t$ is well-defined. We have already somehow seen this fact when we introduced the notion of weak solution, that is, when in Lemma 8.1 we showed that $\mathbf{v} \in W$ implies $\mathbf{v} \in L^{\frac{10}{3}}\left(E_{T}\right)$. Here we have just given a different proof, where the bounds are more clearly pointed out.

Scheffer was the first one to study local regularity for the Navier-Stokes equations. His result states that

Theorem 24.2 ([36]). For $\mathbf{f}=0$, there exists a weak solution of the NavierStokes equations, whose singular set $S$ satisfies

$$
\mathcal{H}^{\frac{5}{3}}(S)<+\infty, \quad \mathcal{H}^{1}(S \cap(E \times\{t\}))<\infty \quad \text { uniformly in } t
$$

where $\mathcal{H}^{k}$ is the Hausdorff $k$-dimensional measure.
Caffarelli, Kohn \& Nirenberg improved the previous result in this way.
Theorem 24.3 ([3]). For any suitable weak solution of the Navier-Stokes equations on an open set in space-time, the associated singular set $S$ satisfies

$$
\mathcal{P}^{1}(S)=0
$$

where $\mathcal{P}^{1}$ is the parabolic 1-dimensional Hausdorff measure.
Such a quantity is analogous but finer than the euclidean 1-dimensional Hausdorff measure. In the sequel, we will explain what we mean by this.

Theorem 24.2 is essentially a consequence of the following.
Proposition 24.1 There are absolute constants $\epsilon_{1} \in(0,1)$ and $c_{1}>0$ such that if $(\mathbf{v}, p)$ is a suitable weak solution in some cylinder $Q_{r} \stackrel{\text { def }}{=} B_{r}\left(x_{o}\right) \times\left(t_{o}-\right.$ $\left.r^{2}, t_{o}\right]$ and

$$
\frac{1}{r^{2}} \iint_{Q_{r}}\left[|\mathbf{v}|^{3}+|\mathbf{v}||p|\right] d x d t+\frac{1}{r^{\frac{13}{4}}} \int_{t_{o}-r^{2}}^{t_{o}}\left(\int_{B_{r}\left(x_{o}\right)}|p| d x\right)^{\frac{5}{4}} d t \leq \epsilon_{1}
$$

then

$$
|\mathbf{v}(x, t)| \leq \frac{c_{1}}{r} \quad \text { in } Q_{\frac{r}{2}}
$$

Hence, in particular, $\mathbf{v}$ is regular.
The Scheffer estimate about the Hausdorff dimension of the singular set comes as a consequence of a covering argument based on negating the main assumption in Proposition 24.1. We will discuss this fact later on.

Theorem 24.3 is essentially a consequence of the following.
Proposition 24.2 There is an absolute constant $\epsilon_{2} \in(0,1)$ such that if $(\mathbf{v}, p)$ is a suitable weak solution in some cylinder $Q_{r} \stackrel{\text { def }}{=} B_{r}\left(x_{o}\right) \times\left(t_{o}-r^{2}, t_{o}\right]$ and

$$
\limsup _{r \rightarrow 0} \frac{1}{r} \iint_{Q_{r}}|\nabla \mathbf{v}|^{2} d x d t \leq \epsilon_{2}
$$

the $|\mathbf{v}|$ is limited.
Again, the estimate on the Hausdorff parabolic dimension of the singular set follows from a proper covering argument based on negating the previous assumption.

## 25 A Compactness Result for Suitable Weak Solutions

We can finally come to the compactness result we mentioned before.
Theorem 25.1. Let $\left\{\left(\mathbf{v}_{n}, p_{n}\right)\right\}$ be a sequence of weak solutions (in the sense of Leray-Hopf) of the Navier-Stokes equations in $Q_{1} \stackrel{\text { def }}{=} B_{1} \times(-1,0]$, such that for some constants $E, E_{o}, E_{1}$ we have

$$
\begin{aligned}
& \int_{B_{1} \times\{t\}}\left|\mathbf{v}_{n}\right|^{2} d x \leq E_{o} \quad \text { for a.e. } t \in(-1,0] \\
& \iint_{Q_{1}}\left|\nabla \mathbf{v}_{n}\right|^{2} d x d t \leq E_{1} \\
& \iint_{Q_{1}}\left|p_{n}\right|^{\frac{3}{2}} d x d t \leq E
\end{aligned}
$$

and the pair $\left(\mathbf{v}_{n}, p_{n}\right)$ satisfies the generalized energy inequality (22.1) for all n. Assume that

$$
\begin{array}{llll}
\mathbf{v}_{n} & \rightharpoonup & \mathbf{v} & \text { weakly in } L^{2}(-1,0 ; V) \\
\mathbf{v}_{n} & \rightharpoonup & \mathbf{v} & \text { weakly }{ }^{*} \text { in } L^{\infty}(-1,0 ; H) \\
p_{n} & \rightharpoonup & p & \text { weakly in } L^{\frac{3}{2}}\left(Q_{1}\right)
\end{array}
$$

Then $(\mathbf{v}, p)$ is a suitable weak solution in $Q_{1}$.

Remark 25.1 The crucial point for the sequence $\left\{\left(\mathbf{v}_{n}, p_{n}\right)\right\}$ is that each pair $\left(\mathbf{v}_{n}, p_{n}\right)$ satisfies the generalized energy inequality.

Under this point of view, we have the following result (for the proof, we refer to [3, Appendix]).
Theorem 25.2. Suppose $E$ is a bounded, open, connected set in $\mathbb{R}^{3}$, locally lying on one side of its boundary, and that $\partial E$ is a smooth manifold. Suppose that $\mathbf{f} \in L^{2}\left(E_{T}\right)$ and $\operatorname{div} \mathbf{f}=0$ (in the weak sense). Finally, assume that $\mathbf{v}_{o} \in H \cap W^{\frac{2}{5}, \frac{5}{4}}(E)$. Then there exists a weak solution $(\mathbf{v}, p)$ of (8.1) in $E_{T}$ satisfying

1. $\mathbf{v} \in L^{2}(0, T ; V) \cap L^{\infty}(0, T ; H)$;
2. $\mathbf{v}(\cdot, t) \rightharpoonup \mathbf{v}_{o}$ weakly in $H$ as $t \rightarrow 0$;
3. $p \in L^{\frac{5}{3}}\left(E_{T}\right)$;
4. For any $\varphi \in C_{o}^{\infty}\left(E_{T}\right), \varphi \geq 0$, $\mathbf{v}$ satisfies (22.1), where we have the extra term $2 \int_{0}^{t} \int_{E} \mathbf{f} \cdot \mathbf{v} d x d t$ on the right-hand side.
Remark 25.2 Notice that 1.-3. say that $(\mathbf{v}, p)$ is a weak solution in the sense of Leray-Hopf, and 4. further specifies that it is a suitable weak solution. On the other hand, we are not saying that all weak solutions are suitable weak solutions, but only that there exists (at least) one suitable weak solution.

Now we prove Theorem 25.1.
Proof. We recall that

$$
\begin{array}{llll}
\mathbf{v}_{n} & \rightharpoonup & \mathbf{v} & \text { weakly in } L^{2}(-1,0 ; V) \\
\mathbf{v}_{n} & \rightharpoonup & \mathbf{v} & \text { weakly* }^{*} \text { in } L^{\infty}(-1,0 ; H), \\
p_{n} & \rightharpoonup & p & \text { weakly in } L^{\frac{3}{2}}\left(Q_{1}\right)
\end{array}
$$

It is enough to prove that $\forall q \in\left[1, \frac{10}{3}\right) \mathbf{v}_{n} \rightarrow \mathbf{v}$ strongly in $L^{q}\left(Q_{1}\right)$. Indeed, in such a case, for any smooth $\varphi \geq 0$, by Fatou's Lemma we have

$$
2 \liminf _{n \rightarrow \infty} \iint_{Q_{1}}\left|\nabla \mathbf{v}_{n}\right|^{2} \varphi d x d t \geq 2 \iint_{Q_{1}}|\nabla \mathbf{v}|^{2} \varphi d x d t
$$

Moreover, by the strong convergence of $\mathbf{v}_{n} \rightarrow \mathbf{v}$ in $L^{3}\left(Q_{1}\right)$ and the weak convergence of $p_{n} \rightarrow p$ in $L^{\frac{3}{2}}\left(Q_{1}\right)$, we conclude about the convergence of the right-hand side of the generalized energy inequality.

In order to show the strong convergence in $L^{q}\left(Q_{1}\right)$, we first prove a proper weak uniform continuity of $\mathbf{v}_{n}$ as a function of time. This is done in the same spirit of what we have done in the proof of Proposition 20.1. As before, we let $Z \stackrel{\text { def }}{=}\left(W_{0}^{2,2}(E)\right)^{\prime}$. By

$$
\partial_{t} \mathbf{v}_{n}-\Delta \mathbf{v}_{n}+\left(\mathbf{v}_{n} \cdot \nabla\right) \mathbf{v}_{n}+\nabla p_{n}=0 \quad \text { weakly in } Q_{1}
$$

and by the weak convergences of $\mathbf{v}_{n}$ to $\mathbf{v}$ and $p_{n}$ to $p$, we can conclude that

$$
\partial_{t} \mathbf{v}_{n} \in L^{\frac{3}{2}}(-1,0 ; Z), \quad\left\|\partial_{t} \mathbf{v}_{n}\right\|_{L^{\frac{3}{2}}(-1,0 ; Z)} \leq c_{o}
$$

for some constant $c_{o}$ which depends on

$$
\sup _{n}\left[\left\|\mathbf{v}_{n}\right\|_{L^{2}(-1,0 ; V)}+\left\|\mathbf{v}_{n}\right\|_{L^{\infty}(-1,0 ; H)}+\left\|p_{n}\right\|_{L^{\frac{3}{2}}\left(Q_{1}\right)}\right]
$$

This allows us to conclude that $\mathbf{v}_{n} \in C^{o}([-1,0] ; Z)$, and also that they are uniformly continuous as functions of $t \in[-1,0]$ with values in $Z$. By an abstract result (see [50, Chapter III, Theorem 2.1]), we can conclude that all $\mathbf{v}_{n}$ stay in a compact subset of $L^{\frac{3}{2}}\left(Q_{1}\right)$. Hence, $\mathbf{v}_{n} \rightarrow \mathbf{v}$ strongly in $L^{\frac{3}{2}}\left(Q_{1}\right)$. Finally, since all $\mathbf{v}_{n}$ are bounded in $L^{\frac{10}{3}}\left(Q_{1}\right)$, we deduce that $\mathbf{v}_{n} \rightarrow \mathbf{v}$ strongly in $L^{q}\left(Q_{1}\right)$ for all $q \in\left[1, \frac{10}{3}\right)$.

## 26 The Partial Regularity Revisited

We give a thorough presentation of the partial regularity theory for NavierStokes equations. We restate Theorems 24.2-24.3 as done in [29].
Theorem 26.1. Let $(\mathbf{v}, p)$ be a suitable weak solution of the Navier-Stokes equations in $Q_{1}$. There exist two positive constants $\epsilon_{o}$ and $c_{o}$, such that, if

$$
\begin{equation*}
\iint_{Q_{1}}\left[|\mathbf{v}|^{3}+|p|^{\frac{3}{2}}\right] d x d t \leq \epsilon_{o} \tag{26.1}
\end{equation*}
$$

then $\mathbf{v}$ is bounded; in particular, $\mathbf{v}$ is $\alpha$-Hölder continuous in $Q_{r}$ for some $\alpha \in(0,1)$ and any $r \in\left(0, \frac{1}{2}\right)$, and $\|\mathbf{v}(x, t)\|_{C^{\alpha}\left(Q_{r}\right)} \leq c_{o}$.

Remark 26.1 At a first reading, it might seem odd, that we jump from boundedness to Hölder continuity, both is space and in time. However, as we know from Theorem 15.1, from the local boundedness of $\mathbf{v}$ one concludes higher regularity in the space variables (here $\mathbf{f}=0$, hence we do not require extra regularity assumptions on it). As pointed out in [3], the effect of the pressure prevents one from proving such a local higher regularity result in the time variable. However, if $\mathbf{v}$ is absolutely continuous in time, and $\mathbf{v}_{t} \in L_{\mathrm{loc}}^{q}(D)$, $q>1$, then the same is true of the space derivatives of $\mathbf{v}_{t}$ on compact subsets of $D$.

Remark 26.2 By the scaling properties discussed in Section 21, (26.1) can be equivalently rewritten as

$$
\frac{1}{r^{2}} \iint_{Q_{r}}\left[|\mathbf{v}|^{3}+|p|^{\frac{3}{2}}\right] d x d t \leq \epsilon_{o}
$$

We omit the proof of Theorem 26.1, even though we will rely on it. The interested reader can refer to [29, Theorem 3.1].

Theorem 26.2. Let $(\mathbf{v}, p)$ be a suitable weak solution of the Navier-Stokes equations in $Q_{1}$. There exists a positive constants $\epsilon_{1}$, such that, if

$$
\begin{equation*}
\limsup _{r \rightarrow 0} \frac{1}{r} \iint_{Q_{r}}|\nabla \mathbf{v}|^{2} d x d t \leq \epsilon_{1} \tag{26.2}
\end{equation*}
$$

then there exist $\theta_{o}, r_{o} \in(0,1)$ and $0<\epsilon_{2} \ll 1$, such that either

$$
\begin{equation*}
\left[A\left(\theta_{o} r\right)\right]^{\frac{3}{2}}+\left[D\left(\theta_{o} r\right)\right]^{2} \leq \frac{1}{2}\left([A(r)]^{\frac{3}{2}}+[D(r)]^{2}\right) \tag{26.3}
\end{equation*}
$$

or

$$
\begin{equation*}
[A(r)]^{\frac{3}{2}}+[D(r)]^{2} \leq \epsilon_{2} \ll 1 \tag{26.4}
\end{equation*}
$$

where $0<r<r_{o}$, and

$$
\begin{equation*}
D(r) \stackrel{\operatorname{def}}{=} \frac{1}{r^{2}} \iint_{Q_{r}}|p|^{\frac{3}{2}} d x d t . \tag{26.5}
\end{equation*}
$$

In the next Sections we proceed in the following way. First, we prove that Theorem 26.2 implies the regularity of $\mathbf{v}$; then we show how negating the main assumption of Theorem 26.2 we obtain an estimate on the Hausdorff parabolic dimension of the singular set $S$. Finally, we give the full proof of Theorem 26.2.

## 27 Theorem 26.2 implies the Regularity of $v$

Let us first suppose that (26.4) holds true. By the interpolation inequality of Lemma 20.1 with $q=\frac{10}{3}$, we have

$$
\begin{aligned}
\iint_{Q_{r}}|\mathbf{v}|^{\frac{10}{3}} d x d t \leq & C r^{\frac{5}{3}}\left[\left(\frac{1}{r} \sup _{-r^{2}<t<0} \int_{B_{r}}|\mathbf{v}|^{2} d x d t\right)^{\frac{2}{3}}\left(\frac{1}{r} \iint_{Q_{r}}|\nabla \mathbf{v}|^{2} d x d t\right)\right. \\
& \left.+\left(\frac{1}{r} \sup _{-r^{2}<t<0} \int_{B_{r}}|\mathbf{v}|^{2} d x\right)^{\frac{5}{3}}\right] \\
= & C r^{\frac{5}{3}}\left[[A(r)]^{\frac{2}{3}} B(r)+[A(r)]^{\frac{5}{3}}\right] .
\end{aligned}
$$

Hence, by (26.2) and (26.4), provided $r_{o}$ is sufficiently small, we have

$$
\iint_{Q_{r}}|\mathbf{v}|^{\frac{10}{3}} d x d t \leq C r^{\frac{5}{3}}\left(\epsilon_{2}^{\frac{4}{9}} \epsilon_{1}+\epsilon_{2}^{\frac{10}{9}}\right)=C r^{\frac{5}{3}} \tilde{\epsilon}
$$

where we have set $\tilde{\epsilon}=\epsilon_{2}^{\frac{4}{9}} \epsilon_{1}+\epsilon_{2}^{\frac{10}{9}}$. By a straightforward application of the Hölder inequality,

$$
\frac{1}{r^{2}} \iint_{Q_{r}}|\mathbf{v}|^{3} d x d t \leq \frac{1}{r^{\frac{3}{2}}}\left[\iint_{Q_{r}}|\mathbf{v}|^{\frac{10}{3}} d x d t\right]^{\frac{9}{10}} \leq C \tilde{\epsilon}
$$

By a possible reduction of $\epsilon_{1}$ and $\epsilon_{2}$, we conclude that

$$
\frac{1}{r^{2}}\left[\int_{Q_{r}}|\mathbf{v}|^{3} d x d t+\iint_{Q_{r}}|p|^{\frac{3}{2}} d x d t\right] \leq \epsilon_{o}
$$

and Remark 26.2 and Theorem 26.1 yield that $\mathbf{v}$ is regular.
On the other hand, if (26.3) holds true, iterating it we obtain that there exists $n_{o} \in \mathbb{N}$ such that

$$
\left[A\left(\theta_{o}^{n_{o}} r\right)\right]^{\frac{3}{2}}+\left[D\left(\theta_{o}^{n_{o}} r\right)\right] \leq \frac{1}{2^{n_{o}}} \leq \epsilon_{2}
$$

and repeating the previous argument, we again conclude that $\mathbf{v}$ is regular.
Remark 27.1 A similar argument is discussed in [30].

## 28 An Estimate of the Hausdorff Parabolic Dimension of the Singular Set

We define the Hausdorff parabolic measure. As above, we let $Q_{r}$ denote the cylinder with radius $r$ and height $r^{2}$. At this stage the upper vertex of the cylinder plays no role.

For any given set $X \subset \mathbb{R}^{3} \times \mathbb{R}$ and $k \geq 0$, we define

$$
\mathcal{P}^{k}=\lim _{\delta \rightarrow 0^{+}} \mathcal{P}_{\delta}^{k}(X)
$$

where

$$
\mathcal{P}_{\delta}^{k}(x)=\inf \left\{\sum_{i=1}^{\infty} r_{i}^{k}: X \subset \bigcup_{i=1}^{\infty} Q_{r_{i}}, r_{i}<\delta\right\}
$$

It is rather easy to see that $\mathcal{P}^{k}$ is an outer measure, for which all Borel sets are measurable; on its $\sigma$-algebra of measurable sets, $\mathcal{P}^{k}$ is a Borel regular measure (we refrain from going into details here).

The Hausdorff measure $\mathcal{H}^{k}$ is defined in an entirely similar manner, but with $Q_{r_{i}}$ replaced by an arbitrary closed set of $\mathbb{R}^{3} \times \mathbb{R}$ of diameter at most $r_{i}$. Typically, one would use balls. Moreover, one usually normalizes $\mathcal{H}^{k}$ for an integer $k$, so that it agrees with the surface area on smooth $k$-dimensional surfaces.

It is not hard to see that $\mathcal{H}^{k} \leq c(k) \mathcal{P}^{k}$.
What we really need is a simple fact: for any $X \subset \mathbb{R}^{3} \times \mathbb{R}, \mathcal{P}^{k}(X)=0$ if and only if, for each $\delta>0$, the set $X$ can be covered by a family of parabolic cylinders $\left\{Q_{r_{i}}\right\}_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} r_{i}^{k}<\delta$.

We also need the following (parabolic) version of Vitali's covering Lemma.

Lemma 28.1 Let $\mathcal{F}$ be any family of parabolic cylinders $Q_{r}(x, t)$ contained in a bounded set of $\mathbb{R}^{3} \times \mathbb{R}$. Then there exists a finite or countable subfamily $\mathcal{F}^{\prime}=\left\{Q_{i}=Q_{r_{i}}\left(x_{i}, t_{i}\right)\right\}$ such that

1. $Q_{i} \cap Q_{j}=\emptyset$ for $i \neq j$;
2. $\forall Q \in \mathcal{F} \exists Q_{r_{i}}\left(x_{i}, t_{i}\right) \in \mathcal{F}^{\prime}$ such that $Q \subset Q_{5 r_{i}}\left(x_{i}, t_{i}\right)$.

We omit the proof. The interested reader can refer to [3, Lemma 6.1]. It is important to remark that

$$
\sup _{Q \in \mathcal{F}} r(Q)<\infty
$$

since we are assuming that $\mathcal{F}$ is contained in a bounded set.
Now let $(\mathbf{v}, p)$ be a suitable weak solution defined in an open set $D$. Without loss of generality, we may assume that $D$ is bounded. By Theorem 26.2, if $(x, t)$ belongs to the singular set $S$,

$$
\underset{r \rightarrow 0}{\limsup } \frac{1}{r} \iint_{Q_{r}}|\nabla \mathbf{v}|^{2} d x d t>\epsilon_{1}
$$

Let $F$ be a neighborhood of $S$ in $D$, and let $\delta>0$. For each $(x, t) \in S$, we choose $Q_{r}(x, t)$ with $r<\delta$ such

$$
\frac{1}{r} \iint_{Q_{r}(x, t)}|\nabla \mathbf{v}|^{2} d y d \tau>\epsilon_{1}, \quad Q_{r}(x, t) \subset F
$$

This plays the role of the family $\mathcal{F}$ of Lemma 28.1. Apply such a lemma to this family of cylinders: we have a disjoint subfamily $\mathcal{F}^{\prime}=\left\{Q_{r_{i}}\left(x_{i}, t_{i}\right)\right\}$ such that

$$
S \subset \bigcup_{i=1}^{\infty} Q_{5 r_{i}}\left(x_{i}, t_{i}\right)
$$

and

$$
\sum_{i=1}^{\infty} r_{i} \leq \frac{1}{\epsilon_{1}} \sum_{i=1}^{\infty} \iint_{Q_{r_{i}}}|\nabla \mathbf{v}|^{2} d y d \tau \leq \frac{1}{\epsilon_{1}} \iint_{F}|\nabla \mathbf{v}|^{2} d y d \tau
$$

Since $\delta$ is arbitrary, we conclude from the previous estimate that $S$ has Lebesgue measure zero, and also that

$$
\mathcal{P}^{1}(S) \leq \frac{5}{\epsilon_{1}} \iint_{F}|\nabla \mathbf{v}|^{2} d y d \tau
$$

for every neighborhood $F$ of $S$. Since $|\nabla \mathbf{v}|^{2}$ is an integrable function, it follows that $\mathcal{P}^{1}(S)=0$.

## 29 Proof of Theorem 26.2: A First Auxiliary Estimate

Let $\theta_{o} \in(0,1)$ the quantity postulated in Theorem 26.2 . For the moment we assume it as given, and we further require it to be in ( $0, \frac{1}{4}$ ). The final argument will determine it. We have the following.

Lemma 29.1 Let $\rho \in(0,1)$ and $r \in\left(\theta_{o} \rho, \frac{\rho}{4}\right)$. Let $(\mathbf{v}, p)$ be a suitable weak solution of the Navier-Stokes equations in $Q_{1}$. Then, for almost every $t \in$ $\left(-\frac{\rho^{2}}{2}, 0\right]$ we have

$$
\begin{aligned}
\frac{1}{r^{2}} \int_{B_{r} \times\{t\}}|p|^{\frac{3}{2}} d x \leq & C_{\theta_{o}} \frac{1}{\rho^{2}} \int_{B_{\rho} \times\{t\}}\left|\mathbf{v}-\overline{\mathbf{v}}_{\rho}\right|^{3} d x \\
& +C_{o}\left(\frac{r}{\rho}\right) \frac{1}{\rho^{2}} \int_{B_{\rho} \times\{t\}}|p|^{\frac{3}{2}} d x
\end{aligned}
$$

where $C_{\theta_{o}}$ is a parameter that depends on $\theta_{o}, C_{o}$ depends only on the dimen$\operatorname{sion} N=3$, and

$$
\overline{\mathbf{v}}_{\rho}=\frac{1}{\left|B_{\rho}\right|} \int_{B_{\rho} \times\{t\}} \mathbf{v} d x
$$

Proof. A rigorous proof is in [24, Lemma 5.3] and [37, Lemma 3.1]. Here we concentrate on the main issues, and sketch the remainder. By rescaling, we can assume $\rho=1$ and directly work with the Navier-Stokes equation in $Q_{1}$. If we take the divergence of both terms, we easily conclude that for all $t \in\left(-\frac{1}{2}, 0\right]$

$$
\Delta p=-\operatorname{div}[(\mathbf{v} \cdot \nabla) \mathbf{v}] \quad \text { in } B_{1}
$$

Since $\operatorname{div} \mathbf{v}=0$, it is a matter of straightforward computations to check that

$$
\operatorname{div}[(\mathbf{v} \cdot \nabla) \mathbf{v}]=\partial_{x_{i}} v_{j} \partial_{x_{j}} v_{i} \quad \text { in } \mathcal{D}^{\prime}\left(B_{1}\right)
$$

with $i, j=1,2,3$. For $t$ fixed, we choose $\bar{\rho} \in\left(\frac{1}{2}, 1\right)$ such that

$$
\int_{\partial B_{\bar{\rho}}}|p|^{\frac{3}{2}} d \sigma \leq 3 \int_{B_{1}}|p|^{\frac{3}{2}} d x
$$

and decompose

$$
p=p_{o}+h
$$

where

$$
\left\{\begin{array} { l } 
{ \Delta p _ { o } = - \partial _ { x _ { i } } v _ { j } \partial _ { x _ { j } } v _ { i } \text { in } B _ { \overline { \rho } } , } \\
{ p _ { o } = 0 \text { on } \partial B _ { \overline { \rho } } , }
\end{array} \quad \left\{\begin{array}{l}
\Delta h=0 \text { in } B_{\bar{\rho}} \\
h=p \text { on } \partial B_{\bar{\rho}}
\end{array}\right.\right.
$$

Thus, for any $\theta \in\left(\theta_{o}, \frac{1}{4}\right)$ we have

$$
\int_{B_{\theta} \times\{t\}}|p|^{\frac{3}{2}} d x \leq C\left[\int_{B_{\theta} \times\{t\}}\left|p_{o}\right|^{\frac{3}{2}} d x+\int_{B_{\theta} \times\{t\}}|h|^{\frac{3}{2}} d x\right] .
$$

We need to estimate the two terms on the right-hand side.
It is not hard to see that the defining relation of $p_{o}$ can be rewritten as

$$
\left\{\begin{array}{l}
\Delta p_{o}=-\partial_{x_{i}}\left(v_{j}-\bar{v}_{j}\right) \partial_{x_{j}}\left(v_{i}-\bar{v}_{i}\right) \text { in } B_{\bar{\rho}} \\
p_{o}=0 \text { on } \partial B_{\bar{\rho}}
\end{array}\right.
$$

where

$$
\bar{v}_{i}=\frac{1}{\left|B_{1}\right|} \int_{B_{1}} v_{i} d x
$$

By the Calderón-Zygmund estimates, we conclude that

$$
\int_{B_{\theta} \times\{t\}}\left|p_{o}\right|^{\frac{3}{2}} d x \leq C_{\theta_{o}} \int_{B_{1} \times\{t\}}|\mathbf{v}-\overline{\mathbf{v}}|^{3} d x
$$

On the other hand, since $h$ is harmonic and $s^{\frac{3}{2}}$ is a convex, monotone increasing function in $[0, \infty),|h|^{\frac{3}{2}}$ is sub-harmonic, so that

$$
\begin{gathered}
|h|^{\frac{3}{2}} \leq \int_{\partial B_{\bar{\rho}}}|p|^{\frac{3}{2}} d \sigma \leq 3 \int_{B_{1}}|p|^{\frac{3}{2}} d x \\
\int_{B_{\theta} \times\{t\}}|h|^{\frac{3}{2}} d x \leq C \int_{B_{1} \times\{t\}}|p|^{\frac{3}{2}} d x
\end{gathered}
$$

and finally

$$
\int_{B_{\theta} \times\{t\}}|p|^{\frac{3}{2}} d x \leq C_{\theta_{o}} \int_{B_{1} \times\{t\}}\left|\mathbf{v}-\overline{\mathbf{v}}_{\rho}\right|^{3} d x+C_{o} \int_{B_{1} \times\{t\}}|p|^{\frac{3}{2}} d x
$$

If we now rescale back to a general $\rho \in(0,1)$, we conclude.
Integrating the previous relation with respect to $t$, and taking into account (26.5) yields

$$
D(r) \leq C_{\theta_{o}} \frac{1}{\rho^{2}} \iint_{Q_{\rho}}\left|\mathbf{v}-\overline{\mathbf{v}}_{\rho}\right|^{3} d x d t+C_{o}\left(\frac{r}{\rho}\right) D(\rho)
$$

for $\rho \in(0,1)$ and $r \in\left(\theta_{o} \rho, \frac{\rho}{4}\right)$. Let us now deal with the first term on the right-hand side. We have

$$
\begin{aligned}
\left.\frac{1}{\rho^{2}} \iint_{Q_{\rho}} \right\rvert\, \mathbf{v} & -\left.\overline{\mathbf{v}}_{\rho}\right|^{3} d x d t=\frac{1}{\rho^{2}} \iint_{Q_{\rho}}\left|\mathbf{v}-\overline{\mathbf{v}}_{\rho}\right|^{\frac{3}{2}}\left|\mathbf{v}-\overline{\mathbf{v}}_{\rho}\right|^{\frac{3}{2}} d x d t \\
& \leq \frac{1}{\rho^{2}} \int_{-\rho^{2}}^{0}\left[\int_{B_{\rho}}\left|\mathbf{v}-\overline{\mathbf{v}}_{\rho}\right|^{2} d x\right]^{\frac{3}{4}}\left[\int_{B_{\rho}}\left|\mathbf{v}-\overline{\mathbf{v}}_{\rho}\right|^{6} d x\right]^{\frac{1}{4}} d t \\
& \leq \frac{1}{\rho^{2}}\left[\sup _{-\rho^{2}<t<0} \int_{B_{\rho}}|\mathbf{v}-\overline{\mathbf{v}}|^{2} d x\right]^{\frac{3}{4}} \int_{-\rho^{2}}^{0}\left[\int_{B_{\rho}}\left|\mathbf{v}-\overline{\mathbf{v}}_{\rho}\right|^{6} d x\right]^{\frac{1}{4}} d t \\
& \leq \frac{C}{\rho^{2}}\left[\frac{1}{\rho} \sup _{-\rho^{2}<t<0} \int_{B_{\rho}}|\mathbf{v}-\overline{\mathbf{v}}|^{2} d x\right]^{\frac{3}{4}} \rho^{\frac{3}{4}}\left[\iint_{Q_{\rho}}|\nabla \mathbf{v}|^{2} d x d t\right]^{\frac{3}{4}} \rho^{\frac{2}{4}}
\end{aligned}
$$

where we have first applied the Sobolev-Poincaré inequality with $p=2, N=3$, $q=6$, and then the Hölder inequality. Hence,

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$$
\begin{aligned}
\frac{1}{\rho^{2}} \iint_{Q_{\rho}}\left|\mathbf{v}-\overline{\mathbf{v}}_{\rho}\right|^{3} d x d t & \leq C\left[\frac{1}{\rho} \sup _{-\rho^{2}<t<0} \int_{B_{\rho}}|\mathbf{v}|^{2} d x\right]^{\frac{3}{4}}\left[\frac{1}{\rho} \iint_{Q_{\rho}}|\nabla \mathbf{v}|^{2} d x d t\right]^{\frac{3}{4}} \\
& =C[A(\rho)]^{\frac{3}{4}}[B(\rho)]^{\frac{3}{4}}
\end{aligned}
$$

and we conclude that for any $\rho \in(0,1)$ and $r \in\left(\theta_{o} \rho, \frac{\rho}{4}\right)$

$$
\begin{equation*}
D(r) \leq C_{\theta_{o}}[A(\rho)]^{\frac{3}{4}}[B(\rho)]^{\frac{3}{4}}+C\left(\frac{r}{\rho}\right) D(\rho) . \tag{29.1}
\end{equation*}
$$

## 30 Proof of Theorem 26.2: A Second Auxiliary Estimate

We consider the localized energy inequality. We take a test function $\varphi \in C_{o}^{\infty}$ such that

$$
\begin{aligned}
& 0 \leq \varphi \leq 1, \quad \varphi=1 \text { in } B_{\frac{\rho}{2}} \times\left(-\frac{\rho^{2}}{2}, 0\right], \quad \varphi=0 \text { in } \mathbb{R}^{3} \backslash\left[B_{\rho} \times\left(-\rho^{2}, \rho^{2}\right]\right] \\
& |\nabla \varphi| \leq \frac{C_{1}}{\rho}, \quad 0 \leq \partial_{t} \varphi \leq \frac{C_{2}}{\rho^{2}}, \quad|\Delta \varphi| \leq \frac{C_{2}}{\rho^{2}}
\end{aligned}
$$

It is easy to see that by the assumptions on $\mathbf{v}$ and $\varphi$, for $t \in\left(-\rho^{2}, 0\right]$ we have

$$
\begin{aligned}
& \int_{B_{\rho} \times\{t\}}|\mathbf{v}|^{2} \varphi d x+2 \iint_{Q_{\rho}}|\nabla \mathbf{v}|^{2} \varphi d x d t \\
& \leq \iint_{Q_{\rho}}|\mathbf{v}|^{2}\left(\partial_{t} \varphi+\Delta \varphi\right) d x d t+\iint_{Q_{\rho}}\left(|\mathbf{v}|^{2}-|\overline{\mathbf{v}}|^{2}+2 p\right) \mathbf{v} \cdot \nabla \varphi d x d t \\
& \leq \iint_{Q_{\rho}}|\mathbf{v}|^{2}\left(\partial_{t} \varphi+\Delta \varphi\right) d x d t+\iint_{Q_{\rho}}\left(\left.| | \mathbf{v}\right|^{2}-|\overline{\mathbf{v}}|^{2} \mid+2 p\right) \mathbf{v} \cdot \nabla \varphi d x d t
\end{aligned}
$$

where

$$
|\overline{\mathbf{v}}|^{2}=\frac{1}{\left|B_{\rho}\right|} \int_{B_{\rho} \times\{t\}}|\mathbf{v}|^{2} d x
$$

We estimate all the terms. We have

$$
\begin{aligned}
\iint_{Q_{\rho}} 2 p \mathbf{v} \cdot \nabla \varphi d x d t & \leq \frac{C_{1}}{\rho} \iint_{Q_{\rho}}|p||\mathbf{v}| d x d t \\
& \leq \frac{C_{1}}{\rho}\left[\iint_{Q_{\rho}}|p|^{\frac{3}{2}} d x d t\right]^{\frac{2}{3}}\left[\iint_{Q_{\rho}}|\mathbf{v}|^{3} d x d t\right]^{\frac{1}{3}} \\
& \leq \frac{C_{1}}{\rho} \rho^{2}\left[\frac{1}{\rho^{2}} \iint_{Q_{\rho}}|p|^{\frac{3}{2}} d x d t\right]^{\frac{2}{3}}\left[\frac{1}{\rho^{2}} \iint_{Q_{\rho}}|\mathbf{v}|^{3} d x d t\right]^{\frac{1}{3}} \\
& =C_{1} \rho[D(\rho)]^{\frac{2}{3}}[C(\rho)]^{\frac{1}{3}}
\end{aligned}
$$

In the same way

$$
\begin{aligned}
&\left.\iint_{Q_{\rho}}| | \mathbf{v}\right|^{2}-|\overline{\mathbf{v}}|^{2} \mid \mathbf{v} \cdot \nabla \varphi d x d t \\
& \leq \frac{C_{1}}{\rho} \int_{-\rho^{2}}^{0}\left[\left.\int_{B_{\rho}}| | \mathbf{v}\right|^{2}-\left.|\overline{\mathbf{v}}|^{2}\right|^{\frac{3}{2}} d x\right]^{\frac{2}{3}}\left[\int_{B_{\rho}}|\mathbf{v}|^{3} d x\right]^{\frac{1}{3}} d t \\
& \leq \frac{C_{1}}{\rho} \int_{-\rho^{2}}^{0}\left[\left.\int_{B_{\rho}}|\nabla| \mathbf{v}\right|^{2}| | d x\right]\left[\int_{B_{\rho}}|\mathbf{v}|^{3} d x\right]^{\frac{1}{3}} d t \\
& \leq \frac{C_{3}}{\rho} \int_{-\rho^{2}}^{0}\left[\int_{B_{\rho}}|\mathbf{v}||\nabla \mathbf{v}| d x\right]\left[\int_{B_{\rho}}|\mathbf{v}|^{3} d x\right]^{\frac{1}{3}} d t \\
& \leq \frac{C_{3}}{\rho} \int_{-\rho^{2}}^{0}\left[\int_{B_{\rho}}|\mathbf{v}|^{2} d x\right]^{\frac{1}{2}}\left[\int_{B_{\rho}}|\nabla \mathbf{v}|^{2} d x\right]^{\frac{1}{2}}\left[\int_{B_{\rho}}|\mathbf{v}|^{3} d x\right]^{\frac{1}{3}} d t \\
& \leq \frac{C_{3}}{\rho}\left[\sup _{-\rho^{2}<t<0} \int_{B_{\rho}}|\mathbf{v}|^{2} d x\right]^{\frac{1}{2}} \cdot \\
& \cdot\left[\iint_{Q_{\rho}}|\nabla \mathbf{v}|^{2} d x d t\right]^{\frac{1}{2}}\left[\int_{-\rho^{2}}^{0}\left(\int_{B_{\rho}}|\mathbf{v}|^{3} d x\right)^{\frac{2}{3}} d t\right]^{\frac{1}{2}}
\end{aligned}
$$

Since

$$
\begin{aligned}
{\left[\int_{-\rho^{2}}^{0}\left(\int_{B_{\rho}}|\mathbf{v}|^{3} d x\right)^{\frac{2}{3}} d t\right]^{\frac{1}{2}} } & \leq\left[\iint_{Q_{\rho}}|\mathbf{v}|^{3} d x d t\right]^{\frac{1}{3}} \rho^{\frac{1}{3}} \\
& \leq\left[\frac{1}{\rho^{2}} \iint_{Q_{\rho}}|\mathbf{v}|^{3} d x d t\right]^{\frac{1}{3}} \rho=\rho[C(\rho)]^{\frac{1}{3}}
\end{aligned}
$$

we conclude that

$$
\begin{aligned}
\left.\iint_{Q_{\rho}}| | \mathbf{v}\right|^{2}-|\overline{\mathbf{v}}|^{2} \mid \mathbf{v} & \cdot \nabla \varphi d x d t \\
\leq & \frac{C_{3}}{\rho}\left[\frac{1}{\rho} \sup _{-\rho^{2}<t<0} \int_{B_{\rho}}|\mathbf{v}|^{2} d x\right]^{\frac{1}{2}}\left[\frac{1}{\rho} \iint_{Q_{\rho}}|\nabla \mathbf{v}|^{2} d x d t\right]^{\frac{1}{2}} . \\
& \cdot \rho^{2}[C(\rho)]^{\frac{1}{3}}=C_{3} \rho[A(\rho)]^{\frac{1}{2}}[B(\rho)]^{\frac{1}{2}}[C(\rho)]^{\frac{1}{3}} .
\end{aligned}
$$

We choose $r \in\left(\theta_{o} \rho, \frac{\rho}{4}\right)$ as in Lemma 29.1; notice that by the previous choices, $\varphi=1$ in $Q_{r}$. We have

$$
\int_{B_{\rho} \times\{t\}}|\mathbf{v}|^{2} \varphi d x+2 \iint_{Q_{\rho}}|\nabla \mathbf{v}|^{2} \varphi d x d t
$$

$$
\begin{aligned}
& \geq \int_{B_{r} \times\{t\}}|\mathbf{v}|^{2} d x+2 \iint_{Q_{r}}|\nabla \mathbf{v}|^{2} d x d t \\
& \geq r\left[\frac{1}{r} \sup _{-r^{2}<t<0} \int_{B_{r} \times\{t\}}|\mathbf{v}|^{2} d x\right]+r\left[\frac{1}{r} \iint_{Q_{r}}|\nabla \mathbf{v}|^{2} d x d t\right] \\
& =r[A(r)+B(r)] .
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
& \iint_{Q_{\rho}}|\mathbf{v}|^{2}\left(\partial_{t} \varphi+\Delta \varphi\right) d x d t \\
& \quad \leq \frac{C_{2}}{\rho^{2}} \iint_{Q_{\rho}}|\mathbf{v}|^{2} d x d t \leq \frac{C_{2}}{\rho^{2}}\left[\iint_{Q_{\rho}}|\mathbf{v}|^{3} d x d t\right]^{\frac{2}{3}} \rho^{\frac{5}{3}} \\
& \quad \leq C_{2}\left[\frac{1}{\rho^{2}} \iint_{Q_{\rho}}|\mathbf{v}|^{3} d x d t\right]^{\frac{2}{3}} \frac{\rho^{\frac{5}{3}}}{\rho^{\frac{2}{3}}}=C_{2} \rho[C(\rho)]^{\frac{2}{3}}
\end{aligned}
$$

Collecting all the estimates obtained so far, yields

$$
\begin{aligned}
& r[A(r)+B(r)] \\
& \quad \leq C_{2} \rho[C(\rho)]^{\frac{2}{3}}+C_{3} \rho[A(\rho)]^{\frac{1}{2}}[B(\rho)]^{\frac{1}{2}}[C(\rho)]^{\frac{1}{3}}+C_{1} \rho[D(\rho)]^{\frac{2}{3}}[C(\rho)]^{\frac{1}{3}}
\end{aligned}
$$

that is

$$
\begin{align*}
A(r) & +B(r) \\
& \leq C\left(\frac{\rho}{r}\right)\left[[C(\rho)]^{\frac{2}{3}}+[A(\rho)]^{\frac{1}{2}}[B(\rho)]^{\frac{1}{2}}[C(\rho)]^{\frac{1}{3}}+[D(\rho)]^{\frac{2}{3}}[C(\rho)]^{\frac{1}{3}}\right] \tag{30.1}
\end{align*}
$$

## 31 The Proof of Theorem 26.2 Concluded

The proof of Theorem 26.2 relies on a clever combination of (23.1), (29.1), (30.1). We recall that $\theta_{o} \in\left(0, \frac{1}{4}\right)$ still needs to be fixed. The next argument will determine it, together with $r_{o}$. We let $\rho=2 r$ and we select three different radii, i.e.

$$
\theta_{o} r, \quad 2 \theta_{o} r, \quad r
$$

so that the three previous relations will be written only in terms of $r$ and $\theta_{o}$. Notice that the choice of the three radii allows us to use all the three (23.1), (29.1), (30.1). Moreover, we remark that

$$
\begin{aligned}
B\left(\theta_{o} r\right) & =\frac{1}{\theta_{o} r} \iint_{Q_{\theta_{o} r}}|\nabla \mathbf{v}|^{2} d x d t \\
& \leq 2 \frac{1}{2 \theta_{o} r} \iint_{Q_{2 \theta_{o} r}}|\nabla \mathbf{v}|^{2} d x d t=2 B\left(2 \theta_{o} r\right)
\end{aligned}
$$

$$
\leq \frac{1}{\theta_{o}} \frac{1}{r} \iint_{Q_{r}}|\nabla \mathbf{v}|^{2} d x d t=\frac{1}{\theta_{o}} B(r)
$$

and

$$
\begin{aligned}
A\left(\theta_{o} r\right) & =\frac{1}{\theta_{o} r} \sup _{-\theta_{o}^{2} r^{2}<t<0} \int_{B_{\theta_{o} r}}|\mathbf{v}|^{2} d x \\
& \leq 2 \frac{1}{2 \theta_{o} r} \sup _{-4 \theta_{o}^{2} r^{2}<t<0} \int_{B_{2 \theta_{o} r}}|\mathbf{v}|^{2} d x=2 A\left(2 \theta_{o} r\right) \\
& \leq \frac{1}{\theta_{o}} \frac{1}{r} \sup _{-r^{2}<t<0} \int_{B_{r}}|\mathbf{v}|^{2} d x=\frac{1}{\theta_{o}} A(r),
\end{aligned}
$$

so that

$$
\begin{align*}
& B\left(\theta_{o} r\right) \leq 2 B\left(2 \theta_{o} r\right) \leq \frac{1}{\theta_{o}} B(r),  \tag{31.1}\\
& A\left(\theta_{o} r\right) \leq 2 A\left(2 \theta_{o} r\right) \leq \frac{1}{\theta_{o}} A(r) .
\end{align*}
$$

Finally

$$
\begin{aligned}
D\left(\theta_{o} r\right) & =\frac{1}{\theta_{o}^{2} r^{2}} \iint_{Q_{\theta_{o} r}}|p|^{\frac{3}{2}} d x d t \\
& \leq 4 \frac{1}{4 \theta_{o}^{2} r^{2}} \iint_{Q_{2 \theta_{o} r}}|p|^{\frac{3}{2}} d x d t=4 D\left(2 \theta_{o} r\right) \\
& \leq \frac{1}{\theta_{o}^{2}} \frac{1}{r^{2}} \iint_{Q_{r}}|p|^{\frac{3}{2}} d x d t=\frac{1}{\theta_{o}^{2}} D(r)
\end{aligned}
$$

that is

$$
\begin{equation*}
D\left(\theta_{o} r\right) \leq 4 D\left(2 \theta_{o} r\right) \leq \frac{1}{\theta_{o}^{2}} D(r) \tag{31.2}
\end{equation*}
$$

Writing (30.1) for $\theta_{o} R$ and $2 \theta_{o} r$ yields

$$
\begin{aligned}
{\left[A\left(\theta_{o} r\right)\right]^{\frac{3}{2}} \leq } & C\left[2^{\frac{3}{2}} C\left(2 \theta_{o} r\right)+2^{\frac{3}{2}}\left[A\left(2 \theta_{o} r\right)\right]^{\frac{3}{4}}\left[B\left(2 \theta_{o} r\right)\right]^{\frac{3}{4}}\left[C\left(2 \theta_{o} r\right)\right]^{\frac{1}{2}}\right. \\
& \left.+2^{\frac{3}{2}}\left[D\left(2 \theta_{o} r\right)\right]\left[C\left(2 \theta_{o} r\right)\right]^{\frac{1}{2}}\right] \\
\leq & C\left[C\left(2 \theta_{o} r\right)+\left[A\left(2 \theta_{o} r\right)\right]^{\frac{3}{2}}\left[B\left(2 \theta_{o} r\right)\right]^{\frac{3}{2}}+\left[D\left(2 \theta_{o} r\right)\right]^{2}\right] .
\end{aligned}
$$

Relying on (23.1), (29.1), and on (31.1)-(31.2) we have

$$
\begin{aligned}
& {\left[A\left(\theta_{o} r\right)\right]^{\frac{3}{2}}+\left[D\left(\theta_{o} r\right)\right]^{2}} \\
& \quad \leq C\left[C\left(2 \theta_{o} r\right)+\left[A\left(2 \theta_{o} r\right)\right]^{\frac{3}{2}}\left[B\left(2 \theta_{o} r\right)\right]^{\frac{3}{2}}+\left[D\left(2 \theta_{o} r\right)\right]^{2}\right] \\
& \quad \leq C\left[\theta_{o}^{3}[A(r)]^{\frac{3}{2}}+\frac{1}{\theta_{o}^{3}}[A(r)]^{\frac{3}{4}}[B(r)]^{\frac{3}{4}}+\left[A\left(2 \theta_{o} r\right)\right]^{\frac{3}{2}}\left[B\left(2 \theta_{o} r\right)\right]^{\frac{3}{2}}\right.
\end{aligned}
$$

$$
\begin{aligned}
& \left.+\theta_{o}^{2}[A(r)]^{\frac{3}{2}}[B(r)]^{\frac{3}{2}}+\theta_{o}^{2}[D(r)]^{2}\right] \\
\leq & C\left[\theta_{o}^{3}[A(r)]^{\frac{3}{2}}+\theta_{o}[A(r)]^{\frac{3}{2}}+\frac{1}{\theta_{o}^{7}}[B(r)]^{\frac{3}{2}}+\frac{1}{\theta_{o}^{\frac{3}{2}}}[A(r)]^{\frac{3}{2}} \frac{1}{\theta_{o}^{\frac{3}{2}}}[B(r)]^{\frac{3}{2}}+\right. \\
& \left.+\theta_{o}^{2}[A(r)]^{\frac{3}{2}}[B(r)]^{\frac{3}{2}}+\theta_{o}^{2}[D(r)]^{2}\right] \\
\leq & C_{1} \theta_{o}\left[[A(r)]^{\frac{3}{2}}\left(1+\theta_{o}^{2}+\theta_{o}[B(r)]^{\frac{3}{2}}+\frac{1}{\theta_{o}^{4}}[B(r)]^{\frac{3}{2}}\right)+[D(r)]^{2}\right] \\
& +\frac{C_{2}}{\theta_{o}^{7}}[B(r)]^{\frac{3}{2}}
\end{aligned}
$$

This holds for $r \in\left(0, r_{o}\right)$ and $\theta_{o} \in\left(0, \frac{1}{4}\right)$, where both $r_{o}$ and $\theta_{o}$ still have to be chosen.

In view of the assumption on $\theta_{o}$ and of the final thesis, without loss of generality we may assume that

$$
\theta_{o}^{2}+\theta_{o}[B(r)]^{\frac{3}{2}}+\frac{1}{\theta_{o}^{4}}[B(r)]^{\frac{3}{2}} \leq \frac{C_{2}}{\theta_{o}^{7}}[B(r)]^{\frac{3}{2}}, \quad \frac{C_{2}}{\theta_{o}^{7}}[B(r)]^{\frac{3}{2}}=\epsilon_{3}
$$

for some $0<\epsilon_{3} \ll 1$ which depends on $\epsilon_{1}$, so that the previous relation becomes

$$
\left[A\left(\theta_{o} r\right)\right]^{\frac{3}{2}}+\left[D\left(\theta_{o} r\right)\right]^{2} \leq C_{3} \theta_{o}\left[[A(r)]^{\frac{3}{2}}+[D(r)]^{2}\right]+\epsilon_{3}
$$

If

$$
[A(r)]^{\frac{3}{2}}+[D(r)]^{2}<\epsilon_{2}
$$

where $\epsilon_{2}$ is the quantity of (26.4), we have finished. Otherwise, let $\epsilon_{3}$ be so small that

$$
\epsilon_{3}=\frac{C_{2}}{\theta_{o}^{7}}[B(r)]^{\frac{3}{2}} \leq C_{3} \theta_{o} \epsilon_{2} \leq C_{3} \theta_{o}\left[[A(r)]^{\frac{3}{2}}+[D(r)]^{2}\right]
$$

whence

$$
\left[A\left(\theta_{o} r\right)\right]^{\frac{3}{2}}+\left[D\left(\theta_{o} r\right)\right]^{2} \leq 2 C_{3} \theta_{o}\left[[A(r)]^{\frac{3}{2}}+[D(r)]^{2}\right]
$$

Now we choose $\theta_{o} \in\left(0, \frac{1}{4}\right)$ so small that $2 C_{3} \theta_{o} \leq \frac{1}{2}$. Once $\theta_{o}$ is determined, from

$$
\frac{C_{2}}{\theta_{o}^{7}}[B(r)]^{\frac{3}{2}} \leq C_{3} \theta_{o} \epsilon_{2}
$$

we choose $r_{o}$ such that the condition on the limsup is satisfied.

## 32 Concluding Remarks

In the following we collect some final remarks about all the results we discussed in these notes.

### 32.1 Partial Regularity

The partial regularity of the Navier-Stokes equations per se is not a strange result. Indeed, in general, for quasi-linear elliptic or parabolic systems the full regularity is not expected. On the other hand, Navier-Stokes equations are a semi-linear system, and in general such a situation ensures better regularity than the quasi-linear setting.

Different (but equivalent!) statements of Theorems 26.1 and 26.2 are collected and briefly commented upon in the survey work [38]. With respect to the original work by Caffarelli, Kohn \& Nirenberg, a somewhat simplified proof of the partial regularity is given in [29]. In these notes, we have mainly followed such a presentation, trying to fix the frequent misprints of Lin's manuscript.

A slight improvement concerning the estimate of the Hausdorff dimension of the singular set $S$ is given in [4].

In [51] Vasseur gives an interesting proof of the partial regularity of suitable weak solutions, which is based on the truncations used by DeGiorgi in his celebrated result about the local Hölder continuity of locally bounded, local weak solutions of linear elliptic equations with bounded and measurable coefficients (see [5]).

### 32.2 Boundary Behavior

Up to now we have said nothing about the smoothness at the boundary. Suppose we consider

$$
B_{r}^{+} \stackrel{\text { def }}{=}\left\{|x|<r, x_{3}>0\right\}, \quad Q_{r}^{+} \stackrel{\text { def }}{=} B_{r}^{+} \times\left(-r^{2}, 0\right]
$$

and we assume

$$
\left.\mathbf{v}\right|_{x_{3}}=0
$$

Can we find reasonable conditions on $\mathbf{v}$ for the space-time origin $(0,0)$ to be a point where the same $\mathbf{v}$ is bounded?

A boundary version of the Ladyzhenskaya-Prodi-Serrin condition can be stated in the following way.

Theorem 32.1. Assume that $\mathbf{v} \in W^{1, n}\left(-1,0 ; W^{2, m}\left(B_{1}^{+}\right)\right) \cap L^{p, q}\left(Q_{1}^{+}\right)$and $p \in L^{n}\left(-1,0 ; W^{1, m}\left(B_{1}^{+}\right)\right)$with

$$
1<m<p, \quad 1<n<q, \quad \frac{3}{p}+\frac{2}{q}=1
$$

is a weak solution of the Navier-Stokes equations in $Q_{1}^{+}$. Moreover, suppose that $\left.\mathbf{v}\right|_{x_{3}}=0$. Then $\mathbf{v}$ is bounded in a neighborhood of the origin.
The result is due to Solonnikov (see [46]).
A boundary version of the partial regularity can be given too. This requires to define what suitable weak solutions at the boundary are. We refrain from going into details here. The interested reader can refer to [38, Section 6].

### 32.3 Suitable Weak Solutions

It is important to point out the real meaning and impact of suitable weak solutions; indeed, it is true that we have localized the energy inequality, and the boundary conditions play no role, but now the pressure $p$ is present in all the estimates, and in more than one sense the pressure can be considered as a substitute for the boundary conditions.

In [19] He gives a proof of the partial regularity result for weak solutions. He starts from a general definition, that contains weak solutions in the sense of Leray-Hopf as a special case.

## Problems and Complements

## 1c Navier-Stokes Equations in Dimensionless Form

A fluid is viscous if its infinitesimal particles at $x$ at time $t$, moving with velocity $\mathbf{v}(x, t)$ encounter a non-zero resistance $\mathbf{R}=-f(|\mathbf{v}|) \mathbf{v}$, where $f$ is a smooth, non-negative function whose form is determined from experiments. For sufficiently slow motions $f(|\mathbf{v}|)=$ const (in the air $|\mathbf{v}| \leq 2 \mathrm{~m} / \mathrm{sec}$ ). In such a case the motion is said to be in viscous regime. For an ideal fluid particle assimilated to a ball of sufficiently small radius $r$

$$
f(|\mathbf{v}|)=6 \pi \mu r \quad \text { for } \quad|\mathbf{v}| \ll 1 \quad \text { (viscous regime) }
$$

where $\mu$ is the dynamic viscosity. This form of $f(|\mathbf{v}|)$ implies that $\mu$ has dimensions $\rho[V][L]$, where $\rho$ is the density of the fluid. The dynamic viscosity is a measure of a resistance offered by a fluid when forced to change its shape. It is a sort of internal friction measured as the resistance elicited by two ideal parallel planes, immersed in the fluid, when forced into a mutual sliding motion. The unit of measure is the poise, after J.L.M. Poiseuille. It is measured in dyne $\cdot \mathrm{s} / \mathrm{cm}^{2}$ and is the force distributed tangentially on a planar surface of $1 \mathrm{~cm}^{2}$, needed to cause a variation of velocity of $1 \mathrm{~cm} / \mathrm{sec}$ between two ideal parallel planes immersed in the fluid and separated by a distance of 1 cm . For water at $20^{\circ} \mathrm{C}$, the dynamic viscosity is .01002 poise. The kinematic viscosity is the ratio of the dynamic viscosity to the density of the fluid. The c.g.s. unit of kinematic viscosity is the stoke, after G. G. Stokes.

For larger speeds, $f(|\mathbf{v}|)$ is proportional to $|\mathbf{v}|$ and the motion is said to be in hydraulic regime (in the air $2 \mathrm{~m} / \mathrm{sec}<|\mathbf{v}| \leq 200 \mathrm{~m} / \mathrm{sec}$ ). For an ideal fluid particle penetrating the fluid and assimilated to a ball of sufficiently small radius $r$

$$
f(|\mathbf{v}|)=5 \pi \mu r^{2}|\mathbf{v}| \quad \text { (hydraulic regime). }
$$

## 4c Non-Homogeneous Boundary Data

Since $E$ is bounded, by the embedding inequalities (2.5)-(2.6), the norm $\|\cdot\|_{V}$ is equivalent to $\|\nabla \cdot\|_{2}$. Thus we regard $V$ as a Hilbert space by the inner product $\langle\cdot, \cdot\rangle=(\nabla \cdot, \nabla \cdot)$. For a fixed pair $(\mathbf{u}, \mathbf{w}) \in V$, let $\mathbf{T}(\mathbf{w}, \mathbf{u})$ be the linear bounded functional in $V$ defined by

$$
\begin{aligned}
\langle T(\mathbf{w}, \mathbf{u}), \boldsymbol{\varphi}\rangle= & \nu \int_{E} \nabla \mathbf{u}: \nabla \boldsymbol{\varphi} d x \\
& -\int_{E}\{\mathbf{w} \cdot(\mathbf{u} \cdot \nabla)+\mathbf{u} \cdot(\mathbf{b} \cdot \nabla)+\mathbf{b} \cdot(\mathbf{u} \cdot \nabla)\} \boldsymbol{\varphi} d x
\end{aligned}
$$

With $\mathbf{g}$ given by (4.4) consider also formally, the functional equation

$$
\begin{equation*}
\mathbf{T}(\mathbf{w}, \mathbf{u})=\mathbf{g} \quad \text { in } V^{*} \tag{4.1c}
\end{equation*}
$$

Then a weak solution of (4.1) is an element $\mathbf{u} \in V$ such that $\mathbf{T}(\mathbf{u}, \mathbf{u})=\mathbf{g}$.
Proposition 4.1c Let the assumptions on $\mathbf{f}$ and $\mathbf{a}$ be in force so that in particular (4.9) holds. Then for all $\mathbf{w}, \mathbf{u} \in V$ with $\|\mathbf{u}\|_{V}>0$

$$
\begin{equation*}
\|\mathbf{T}(\mathbf{w}, \mathbf{u})\| \geq \frac{1}{2} \nu \tag{4.2c}
\end{equation*}
$$

Moreover, for any fixed $\mathbf{w} \in V$, any solution $\mathbf{u} \in V$ of (4.1c) satisfies

$$
\begin{equation*}
\|\nabla \mathbf{u}\|_{2} \leq \frac{2 \gamma}{\nu}\left[\|\mathbf{f}\|_{\frac{6}{5}}+\nu\|\nabla \mathbf{b}\|_{2}+\|\mathbf{b}\|_{4}^{2}\right] \tag{4.3c}
\end{equation*}
$$

where $\gamma$ is the constant of the embedding of $V$ into $L^{6}\left(E ; \mathbb{R}^{3}\right)$.
Remark 4.1c These estimates are independent of $\mathbf{w}$. Thus in particular they hold for solutions of $\mathbf{T}(\mathbf{u}, \mathbf{u})=\mathbf{g}$.

Proof.

$$
\|\mathbf{T}(\mathbf{u}, \mathbf{w})\|=\sup _{\|\boldsymbol{\varphi}\|=1}\langle\mathbf{T}(\mathbf{u}, \mathbf{w}), \boldsymbol{\varphi}\rangle \geq \frac{\langle\mathbf{T}(\mathbf{u}, \mathbf{u}), \mathbf{u}\rangle}{\|\mathbf{u}\|_{V}}
$$

## 4.1c Solving (4.1) by Galerkin Approximations

The space $V$ is a separable Hilbert space by the inner product $(\nabla \cdot, \nabla \cdot)$ and hence it admits a countable base $\left(\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}, \ldots\right)$, orthonormal in $(\nabla \cdot, \nabla \cdot)$. Setting $V_{n}=\operatorname{span}\left\{\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}\right\}$, every $\mathbf{w} \in V$ can be written as

$$
\begin{equation*}
\mathbf{w}=\mathbf{w}_{n}+\sum_{j>n} w_{j} \mathbf{e}_{j} \quad \text { where } \quad \mathbf{w}_{n}=\sum_{j=1}^{n} w_{j} \mathbf{e}_{j} \in V_{n} \tag{4.4c}
\end{equation*}
$$

for scalar $w_{j}$. If $\mathbf{u} \in V$ is a solution of (4.1) in the sense of (4.4)-(4.5), the latter holds for $\boldsymbol{\varphi}=\mathbf{e}_{i}$. In the resulting expression write $\mathbf{u}$ in the form (4.4c), and
notice that the terms involving $\sum_{j>n} u_{j} \mathbf{e}_{j}$ tend to zero as $n \rightarrow \infty$. This suggests defining an approximate solution of (4.1) a function $\mathbf{u}_{n} \in V_{n}$, satisfying (4.5) for $\varphi=\mathbf{e}_{i}$, for all $i=1, \ldots, n$, i.e.,

$$
\begin{aligned}
\sum_{j=1}^{n} & \left\{\nu \int_{E} \nabla \mathbf{e}_{j}: \nabla \mathbf{e}_{i} d x+\int_{E} \mathbf{e}_{i} \cdot\left(\mathbf{u}_{n} \cdot \nabla\right) \mathbf{e}_{j} d x\right. \\
& \left.+\int_{E} \mathbf{e}_{i} \cdot(\mathbf{b} \cdot \nabla) \mathbf{e}_{j} d x+\int_{E} \mathbf{e}_{i} \cdot\left(\mathbf{e}_{j} \cdot \nabla\right) \mathbf{b} d x\right\}_{i j} u_{j}=\int_{E} \mathbf{g} \cdot \mathbf{e}_{i} d x
\end{aligned}
$$

The elements $\{\cdots\}_{i j}$ are the entries of a $n \times n$ matrix $\left(T_{i j}\left(\mathbf{u}_{n}\right)\right)$. The right hand side defines a vector $\left(g_{1}, \ldots, g_{n}\right) \in \mathbb{R}^{n}$, identified with $\mathbf{g}_{n} \in V_{n}$. More generally, for $\mathbf{w}_{n} \in V_{n}$ define $T_{i j}\left(\mathbf{w}_{n}\right)$ as $T_{i j}\left(\mathbf{u}_{n}\right)$, with $\mathbf{w}_{n}$ replacing $\mathbf{u}_{n}$, and seek solutions $\left(u_{1}, \ldots, u_{n}\right) \in \mathbb{R}^{n}$ of

$$
\begin{equation*}
T_{i j}\left(\mathbf{w}_{\mathbf{n}}\right) u_{j}=g_{i} \quad \text { for } i=1, \ldots, n \tag{4.5c}
\end{equation*}
$$

The corresponding $\mathbf{u}_{n} \in V_{n}$ is a solution of $\mathbf{T}\left(\mathbf{w}_{n}, \mathbf{u}_{n}\right)=\mathbf{g}_{n}$. The Galerkin approximations of (4.1) is function $\mathbf{u}_{n} \in V_{n}$ satisfying

$$
\begin{equation*}
\left\langle\mathbf{T}\left(\mathbf{u}_{\mathbf{n}}, \mathbf{u}_{\mathbf{n}}\right), \boldsymbol{\varphi}_{n}\right\rangle=\left\langle\mathbf{g}_{n}, \boldsymbol{\varphi}_{n}\right\rangle \quad \text { for all } \boldsymbol{\varphi}_{n} \in V_{n} \tag{4.6c}
\end{equation*}
$$

## Proposition 4.2c

(i). For all $n$ there exists a Galerkin approximation $\mathbf{u}_{n}$ to (4.1).
(ii). A sequence $\left\{\mathbf{u}_{n}\right\}$ of Galerkin approximations is equibounded in $V$.
(iii). Any $\mathbf{u}$ in the weak closure of $\left\{\mathbf{u}_{n}\right\}$ is a solution of (4.1).

Prove the proposition by the following steps:
Step 1. Use (4.2c) to prove that $\operatorname{det}\left(T_{i j}\left(\mathbf{w}_{n}\right)\right) \geq \frac{1}{2} \nu$, for all $\mathbf{w}_{n} \in V_{n}$. Therefore, for all $\mathbf{g}_{n} \in V_{n}$ there exists a unique $\mathbf{u}_{n} \underset{\sim}{\in} V_{n}$ satisfying ( 4.5 c ).
Step 2. Introduce the map $\mathbf{B}\left(\mathbf{w}_{n}\right)=\mathbf{u}_{n}$ from $\mathbb{R}^{n}$ into itself. Prove that such a map and its inverse $\mathbf{B}^{-1}$ are well defined and continuous in $\mathbb{R}^{n}$.
Step 3. Use (4.3c) to prove that map $\mathbf{B}^{-1}$ maps the ball of radius $2 \gamma\left[\|\mathbf{f}\|_{\frac{6}{5}}+\nu\|\nabla \mathbf{b}\|_{2}+\|\mathbf{b}\|_{4}^{2}\right] / \nu$ into itself.
Step 4. Therefore, $\mathbf{B}(\cdot)$ has a fixed point by the Brouwer fixed point theorem (see [2]). Any such fixed point, identified with an element $\mathbf{u}_{n} \in V_{n}$, solves (4.6c).

Step 5. Use (4.3c) to prove that $\left\|\nabla \mathbf{u}_{n}\right\|_{2} \leq 2 \gamma\left[\|\mathbf{f}\|_{\frac{6}{5}}+\nu\|\nabla \mathbf{b}\|_{2}+\|\mathbf{b}\|_{4}^{2}\right] / \nu$ for all $n \in \mathbb{N}$. Therefore, the embedding $\left\{\mathbf{u}_{n}\right\} \subset L^{p}\left(E ; \mathbb{R}^{3}\right)$ is compact for all $1<p<6$.
Step 6. Having fixed $\mathbf{u}$ in the weak closure of $\left\{\mathbf{u}_{n}\right\}$, a subsequence can be selected and relabeled with $n$, such that $\left\{\nabla \mathbf{u}_{n}\right\} \rightarrow \nabla \mathbf{u}$ weakly in $L^{2}\left(E ; \mathbb{R}^{3}\right)$ and $\left\{\mathbf{u}_{n}\right\} \rightarrow \mathbf{u}$ strongly in $L^{4}\left(E ; \mathbb{R}^{3}\right)$.
Step 7. Let $n \rightarrow \infty$ in (4.6c), justifying the limits of each term, to establish the existence of a solution of (4.1) in the form (4.5).

## 4.2c Extending Fields a $\in W^{\frac{1}{2}, 2}\left(\partial E ; \mathbb{R}^{3}\right)$, Satisfying (4.2) into Solenoidal Fields $b \in W^{1,2}\left(E ; \mathbb{R}^{3}\right)$

We will prove the following result.
Proposition 4.3c Let $E$ be a bounded, simply connected, open set in $\mathbb{R}^{N}$ ( $N=2,3$ ) with boundary $\partial E$ of class $C^{1}$ having one connected component, and satisfying the segment property. For every vector field $\mathbf{a} \in W^{\frac{1}{2}, 2}\left(\partial E ; \mathbb{R}^{N}\right)$ satisfying

$$
\int_{\partial E} \mathbf{a} \cdot \mathbf{n} d \sigma=0
$$

where $\mathbf{n}$ is the outward unit normal to $\partial E$, there exists a vector field $\boldsymbol{\psi} \in$ $W^{2,2}\left(E ; \mathbb{R}^{N}\right)$ such that $\mathbf{b}=\operatorname{curl} \boldsymbol{\psi}$ is an extension of $\mathbf{a}$ into $E$. The function $\psi$ can be chosen to be compactly supported about $\partial E$. Furthermore, for every fixed $\epsilon>0$ the vector field $\boldsymbol{\psi}$ can be chosen so that for every $\mathbf{u} \in V$

$$
\begin{equation*}
\left\|\left|\mathbf{u}\|\operatorname{curl} \boldsymbol{\psi} \mid\|_{2} \leq \chi(\epsilon)\|\nabla \mathbf{u}\|_{2} \quad \text { in } E\right.\right. \tag{4.7c}
\end{equation*}
$$

where $\chi(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Finally if $\mathbf{a} \in C^{k}\left(\partial E ; \mathbb{R}^{N}\right)$, for some $k=1, \ldots$, and $\partial E$ is of class $C^{k+1}$, then $\boldsymbol{\psi}$ can be taken of class $C^{k+1}\left(E ; \mathbb{R}^{N}\right)$.

We need some preliminary Lemmas. The first and the second ones are taken from [15], Chapter III, Section 6. In the last one, we follow the approach developed in [23], Chapter 1, Section 2 and in [13], Lemma 2.1; see also [16], Chapter VIII, Section 4.

Lemma 4.1c Let $E$ be a bounded, open set in $\mathbb{R}^{N}$ and let

$$
\delta(x)=\operatorname{dist}(x, \partial E)
$$

For any $\epsilon>0$ define $\gamma(\epsilon)=\exp \left(-\frac{1}{\epsilon}\right)$. Then, there exists a function $\varphi_{\epsilon} \in$ $C^{\infty}(\bar{E})$ such that

- $\left|\varphi_{\epsilon}(x)\right| \leq 1$ for all $x \in E$,
- $\varphi_{\epsilon}(x)=1$ if $\delta(x)<\gamma^{2}(\epsilon) /\left(2 \kappa_{1}\right)$,
- $\varphi_{\epsilon}(x)=0$ if $\delta(x) \geq 2 \gamma(\epsilon)$,
- $\left|\nabla \varphi_{\epsilon}(x)\right| \leq \kappa_{2} \epsilon / \delta(x)$ for all $x \in E$,
where $\kappa_{1}, \kappa_{2}$ depend only on $N$.
Proof. We first recall the following result, for whose proof we refer to [47], Chapter VI, Theorem 2:

There exists a function $\rho \in C^{\infty}(E)$ such that for all $x \in E$

1. $\delta(x) \leq \rho(x)$;
2. for any partial derivative of order $\alpha,|\alpha| \geq 0$, we have

$$
\left|D^{\alpha} \rho(x)\right| \leq \kappa_{|\alpha|+1}[\delta(x)]^{1-|\alpha|}
$$

where all $\kappa_{|\alpha|+1}$ depend only on $\alpha$ and $N$.
Now consider the function $\xi_{\epsilon}: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$
\xi_{\epsilon}(t)= \begin{cases}1 & \text { if } t<\gamma^{2}(\epsilon) \\ \epsilon \ln \left(\frac{\gamma(\epsilon)}{t}\right) & \text { if } \gamma^{2}(\epsilon)<t<\gamma(\epsilon) \\ 0 & \text { if } t>\gamma(\epsilon)\end{cases}
$$

Now, choose $\eta=\gamma^{2}(\epsilon) / 2$, a mollifier $j_{\eta}$, and consider the mollified function $\Xi_{\epsilon} \equiv \xi_{\epsilon} * j_{\eta}$. It is not hard to check that

- $\Xi_{\epsilon}(t)=1$ for $t<\gamma^{2}(\epsilon) / 2$,
- $\Xi_{\epsilon}(t)=0$ for $t>2 \gamma(\epsilon)$,
- $\left|\Xi_{\epsilon}(t)\right| \leq 1$ for all $t \in \mathbb{R}$,
- $\left|\Xi_{\epsilon}^{\prime}(t)\right| \leq \epsilon / t$ for all $t \in \mathbb{R}$.

We now let $\varphi_{\epsilon}(x)=\Xi_{\epsilon}(\rho(x))$; taking into account 1. and 2. above and the bound on $\left|\Xi_{\epsilon}^{\prime}\right|$, we conclude that

$$
\begin{aligned}
\varphi_{\epsilon}(x)=1 & \text { if } \delta(x)<\gamma^{2}(\epsilon) / 2 \kappa_{1} \\
\varphi_{\epsilon}(x)=0 & \text { if } \delta(x)>2 \gamma(\epsilon) \\
\left|\nabla \varphi_{\epsilon}(x)\right| \leq \kappa_{2} \epsilon / \rho(x) \leq \kappa_{2} \epsilon / \delta(x) & \text { for all } x \in E
\end{aligned}
$$

for proper, positive $\kappa_{1}$ and $\kappa_{2}$, which depend only on $N$.
Lemma 4.2c Let $E \subset \mathbb{R}^{N}$ be a bounded, Lipschitz, open set. Then, there exists $c=c(E)$ such that for all $u \in W_{o}^{1,2}(E)$ we have

$$
\left\|\frac{u}{\delta}\right\|_{2} \leq c\|\nabla u\|_{2}
$$

where $\delta=\delta(x)$ is the function, which has been defined above.
Proof. By density it suffices to assume $u \in C_{o}^{\infty}(E)$. By the theory of Sobolev spaces, for every open set $E^{\prime} \subset \subset E$, we have

$$
\|u\|_{2 ; E^{\prime}} \leq c_{1}\|\nabla u\|_{2 ; E}
$$

where $c_{1}=c_{1}(N, E)$. In order to conclude, we have to take into account the behavior close to the boundary $\partial E$. We recall that a bounded domain $E \subset \mathbb{R}^{N}$ is said to be a Lipschitz domain, if there exists a radius $r_{o}$, such that for each $y \in \partial E$, in an appropriate coordinate system,

$$
\begin{aligned}
E \cap B_{8 r_{o}}(y) & =\left\{x=\left(x^{\prime}, x_{N}\right) \in \mathbb{R}^{N}: x_{N}>\Phi\left(x^{\prime}\right)\right\} \cap B_{8 r_{o}}(y), \\
\partial E \cap B_{8 r_{o}}(y) & =\left\{x=\left(x^{\prime}, x_{N}\right) \in \mathbb{R}^{N}: x_{N}=\Phi\left(x^{\prime}\right)\right\} \cap B_{8 r_{o}}(y),
\end{aligned}
$$

where $\Phi$ is a Lipschitz function, with $\|\nabla \Phi\|_{L^{\infty}} \leq L$. The quantities $r_{o}$ and $L$ are independent of $y \in \partial E$. We say that $L$ is the Lipschitz constant of $E$.

If we set

$$
G(y)=E \cap B_{8 r_{o}}(y)
$$

it is not hard to check that $\exists c_{2}=c_{2}(E)$, such that

$$
\forall x \in G(y) \quad x_{N}-\Phi\left(x^{\prime}\right) \leq c_{2} \delta(x)
$$

Therefore, if we let $y=\left(y^{\prime}, y_{N}\right)$, and $B_{r_{o}}^{\prime}\left(y^{\prime}\right)$ denotes the $(N-1)$-dimensional ball centered at $y^{\prime}$, we have

$$
\int_{G(y)} \frac{1}{\delta^{2}(x)}|u(x)|^{2} d x \leq c_{2} \int_{B_{8 r_{o}}^{\prime}\left(y^{\prime}\right)} d x^{\prime} \int_{\Phi\left(x^{\prime}\right)}^{\Phi\left(x^{\prime}\right)+16 r_{o}} \frac{\left|u\left(x^{\prime}, x_{N}\right)\right|^{2}}{\left|x_{N}-\Phi\left(x^{\prime}\right)\right|^{2}} d x_{N}
$$

and the wanted estimate follows from the one-dimensional inequality

$$
\int_{0}^{\infty} \frac{|h(t)|^{q}}{t^{q}} d t \leq \frac{q}{q-1} \int_{0}^{\infty}\left|\frac{d h}{d t}\right|^{q} d t
$$

which holds for any $h \in C_{o}^{\infty}\left(\mathbb{R}_{+}\right)$and for any $q>1$, and which can be easily proved integrating the identity

$$
\frac{|h(t)|^{q}}{t^{q}}=\frac{d}{d t}\left[\frac{t^{1-q}}{1-q}|h(t)|^{q}\right]-\frac{t^{1-q}}{1-q} \frac{d}{d t}|h(t)|^{q} .
$$

Lemma 4.3c Let $E$ be a bounded, simply connected, open set in $\mathbb{R}^{N}(N=$ $2,3)$ with boundary $\partial E$ of class $C^{1}$, having one connected component, and satisfying the segment property. For every vector field $\mathbf{a} \in W^{\frac{1}{2}, 2}\left(\partial E ; \mathbb{R}^{N}\right)$ satisfying

$$
\begin{equation*}
\int_{\partial E} \mathbf{a} \cdot \mathbf{n} d \sigma=0 \tag{4.8c}
\end{equation*}
$$

where $\mathbf{n}$ is the outward unit normal to $\partial E$, there exists a vector field $\mathbf{w} \in$ $W^{2,2}\left(E ; \mathbb{R}^{N}\right)$ such that $\mathbf{a}=$ curl $\mathbf{w}$ in the sense of traces on $\partial E$. Moreover,

$$
\begin{equation*}
\|\mathbf{w}\|_{W^{2,2}(E)} \leq c\|\mathbf{a}\|_{W^{1 / 2,2}(\partial E)}, \tag{4.9c}
\end{equation*}
$$

where $c$ depends on $N$ and $E$.
Proof. For the moment we assume $\partial E$ smooth, without further specification, to the extent that all the necessary operations can be performed. At the end we will briefly discuss how conditions can be relaxed in a way that all the needed estimates are still justified.

First we consider the case $N=3$; later on we will briefly deal with $N=2$, which is considerably simpler. Let $\mathbf{n}$ be the outward unit normal to $\partial E$ and rewrite a as

$$
\mathbf{a}=\mathbf{a}_{\tau}+a_{n} \mathbf{n}
$$

where $a_{n}=\mathbf{a} \cdot \mathbf{n}$ and $\mathbf{a}_{\tau}$ is the component of $\mathbf{a}$ tangential to $\partial E$.

We first look for a solenoidal vector field $\mathbf{b}_{1}: E \rightarrow \mathbb{R}^{3}, \mathbf{b}_{1} \in W^{1,2}\left(E ; \mathbb{R}^{3}\right)$, such that

$$
\left\{\begin{aligned}
& \mathbf{b}_{1}=\nabla \varphi \quad \text { in } E \\
& \mathbf{b}_{1} \cdot \mathbf{n}=a_{n} \quad \text { on } \partial E
\end{aligned}\right.
$$

Since $\operatorname{div} \mathbf{b}_{1}=0$, this implies that $\varphi$ is a solution of

$$
\left\{\begin{array}{l}
\Delta \varphi=0 \quad \text { in } \quad E, \\
\frac{\partial \varphi}{\partial n}=a_{n} \quad \text { on } \quad \partial E .
\end{array}\right.
$$

This is a Neumann problem for the Laplacean in $E$, and condition (4.8c) ensures that a solution $\varphi \in W^{2,2}(E)$ exists, up to an arbitrary constant. Hence $\mathbf{b}_{1} \in W^{1,2}\left(E ; \mathbb{R}^{3}\right)$ is well-defined. Moreover, since $E$ is simply connected, by well-known results, there exists $\mathbf{w}_{1} \in W^{2,2}\left(E ; \mathbb{R}^{3}\right)$ such that

$$
\mathbf{b}_{1}=\operatorname{curl} \mathbf{w}_{1}
$$

If we now let

$$
\mathbf{b} \stackrel{\text { def }}{=} \mathbf{b}_{1}+\mathbf{b}_{2}
$$

the vector $\mathbf{b}$ is completely determined, if $\mathbf{b}_{2}$ solves

$$
\left\{\begin{align*}
\operatorname{div} \mathbf{b}_{2} & =0 \quad \text { in } E  \tag{4.10c}\\
\mathbf{b}_{2} & =\mathbf{a}-\mathbf{b}_{1} \quad \text { on } \partial E
\end{align*}\right.
$$

taking into account that, by construction,

$$
\begin{equation*}
\left(\mathbf{a}-\mathbf{b}_{1}\right) \cdot \mathbf{n}=0 \quad \text { on } \quad \partial E . \tag{4.11c}
\end{equation*}
$$

In order to explain the main ideas underlying the construction of $\mathbf{b}_{2}$, we first consider the simple situation of

$$
E=\left\{\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{R}^{3}: x_{3}>0\right\}, \partial E=\left\{\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{R}^{3}: x_{3}=0\right\}
$$

before dealing with general $E$ and $\partial E$. At this step $E$ is not bounded, but it is immaterial for what we are going to do. We have

$$
\begin{aligned}
\mathbf{a} & =a_{1}\left(x_{1}, x_{2}\right) \mathbf{e}_{1}+a_{2}\left(x_{1}, x_{2}\right) \mathbf{e}_{2}+a_{3}\left(x_{1}, x_{2}\right) \mathbf{e}_{3} \\
\mathbf{b}_{1} & =b_{1,1}\left(x_{1}, x_{2}, x_{3}\right) \mathbf{e}_{1}+b_{1,2}\left(x_{1}, x_{2}, x_{3}\right) \mathbf{e}_{2}+b_{1,3}\left(x_{1}, x_{2}, x_{3}\right) \mathbf{e}_{3}
\end{aligned}
$$

where $b_{1,3}\left(x_{1}, x_{2}, 0\right)=a_{3}\left(x_{1}, x_{2}\right)$. By (4.11c), we have

$$
\mathbf{a}-\left.\mathbf{b}_{1}\right|_{\partial E}=\left(a_{1}\left(x_{1}, x_{2}\right)-b_{1,1}\left(x_{1}, x_{2}, 0\right)\right) \mathbf{e}_{1}+\left(a_{2}\left(x_{1}, x_{2}\right)-b_{1,2}\left(x_{1}, x_{2}, 0\right)\right) \mathbf{e}_{2} .
$$

If we let

$$
h_{i}\left(x_{1}, x_{2}\right)=a_{i}\left(x_{1}, x_{2}\right)-b_{1, i}\left(x_{1}, x_{2}\right) \text { for } i=1,2, \quad h_{3}\left(x_{1}, x_{2}\right)=0
$$

and $\mathbf{b}_{2}=$ curl $\mathbf{w}_{2}$, solving (4.10c) reduces to determining $\mathbf{w}_{2}: E \rightarrow \mathbb{R}^{3}$, $\mathbf{w}_{2} \in W^{2,2}\left(E ; \mathbb{R}^{3}\right)$, such that $\left.\operatorname{curl} \mathbf{w}_{2}\right|_{x_{3}=0}=\mathbf{h}$ in the sense of traces, i.e.

$$
\begin{aligned}
\frac{\partial w_{2,3}}{\partial x_{2}}\left(x_{1}, x_{2}, 0\right)-\frac{\partial w_{2,2}}{\partial x_{3}}\left(x_{1}, x_{2}, 0\right) & =h_{1}\left(x_{1}, x_{2}\right) \\
\frac{\partial w_{2,1}}{\partial x_{3}}\left(x_{1}, x_{2}, 0\right)-\frac{\partial w_{2,3}}{\partial x_{1}}\left(x_{1}, x_{2}, 0\right) & =h_{2}\left(x_{1}, x_{2}\right) \\
\frac{\partial w_{2,2}}{\partial x_{1}}\left(x_{1}, x_{2}, 0\right)-\frac{\partial w_{2,1}}{\partial x_{2}}\left(x_{1}, x_{2}, 0\right) & =0 .
\end{aligned}
$$

Choosing

$$
\frac{\partial w_{2,3}}{\partial x_{1}}\left(x_{1}, x_{2}, 0\right)=\frac{\partial w_{2,3}}{\partial x_{2}}\left(x_{1}, x_{2}, 0\right)=\frac{\partial w_{2,2}}{\partial x_{1}}\left(x_{1}, x_{2}, 0\right)=\frac{\partial w_{2,1}}{\partial x_{2}}\left(x_{1}, x_{2}, 0\right)=0
$$

yields

$$
\begin{aligned}
-\frac{\partial w_{2,2}}{\partial x_{3}}\left(x_{1}, x_{2}, 0\right) & =h_{1}\left(x_{1}, x_{2}\right) \\
\frac{\partial w_{2,1}}{\partial x_{3}}\left(x_{1}, x_{2}, 0\right) & =h_{2}\left(x_{1}, x_{2}\right)
\end{aligned}
$$

If we assume that $w_{2,1}\left(x_{1}, x_{2}, 0\right)=w_{2,2}\left(x_{1}, x_{2}, 0\right)=w_{2,3}\left(x_{1}, x_{2}, 0\right)=0$, we conclude that a solution is given by

$$
\begin{aligned}
& w_{2,1}\left(x_{1}, x_{2}, x_{3}\right)=x_{3} h_{2}\left(x_{1}, x_{2}\right) \\
& w_{2,2}\left(x_{1}, x_{2}, x_{3}\right)=-x_{3} h_{1}\left(x_{1}, x_{2}\right) \\
& w_{2,3}\left(x_{1}, x_{2}, x_{3}\right)=0
\end{aligned}
$$

Notice that $\mathbf{w}_{2}\left(x_{1}, x_{2}, 0\right)=0$ for any $\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2}$. The vector field we were looking for is then

$$
\mathbf{b}=\operatorname{curl} \mathbf{w}_{1}+\operatorname{curl} \mathbf{w}_{2}=\operatorname{curl}\left(\mathbf{w}_{1}+\mathbf{w}_{2}\right)=\operatorname{curl} \mathbf{w} .
$$

Now, we turn to consider the case of a general simply connected, bounded open set $E \subset \mathbb{R}^{3}$, with smooth boundary $\partial E$ having one connected component. As before, we can proceed with the construction of the vector field $\mathbf{b}_{1}=\nabla \varphi=$ $\operatorname{curl} \mathbf{w}_{1}$, so that it only remains to determine $\mathbf{w}_{2}$ in this new context.

Consider a partition of the unity for the set $E$, namely a collection of $C^{\infty}$ functions $\psi_{k}$ with compact support $\Delta_{k}$, such that

$$
\sum_{k} \psi_{k}(x)=1 \quad \forall x \in E .
$$

Without loss of generality, we can assume each $\psi_{k}$ to be defined on all $\mathbb{R}^{3}$. Let $\partial E_{k}$ be the intersection of $\partial E$ with the domain where $\psi_{k} \not \equiv 0$, provided such a domain has indeed a non-empty intersection with $\partial E$. For each fixed $\psi_{k}$, we can now introduce a smooth change of variables $\left(y_{1, k}, y_{2, k}, y_{3, k}\right)$ such that in
the new coordinates, $\partial E_{k}$ is the graph of $y_{3}=0$ in a compact set $D_{k} \subset \mathbb{R}^{2}$, and the coordinate system is orthogonal on $\partial E_{k}$.

If we let $\left(\mathbf{a}-\mathbf{b}_{1}\right)_{k}=\psi_{k}\left(\mathbf{a}-\mathbf{b}_{1}\right)$, we are going to build a vector field $\left(\mathbf{b}_{2}\right)_{k}=\operatorname{curl}\left(\mathbf{w}_{2}\right)_{k}$ such that $\left(\mathbf{b}_{2}\right)_{k}=\left(\mathbf{a}-\mathbf{b}_{1}\right)_{k}$ on $\partial E$, and

$$
\mathbf{b}=\operatorname{curl} \mathbf{w}_{1}+\sum_{k}\left(\mathbf{b}_{2}\right)_{k}=\operatorname{curl} \mathbf{w}_{1}+\sum_{k} \operatorname{curl}\left(\mathbf{w}_{2}\right)_{k} .
$$

Take $\left(\mathbf{a}-\mathbf{b}_{1}\right)_{k}$, perform the previously mentioned change of variables that flattens the portion $\partial E_{k}$ of the boundary $\partial E$, and let $\mathbf{h}_{k}$ be the restriction on $y_{3}=0$ of the new vector field thus obtained. By construction, $\mathbf{h}_{k}$ has compact support, and we also have $\mathbf{h}_{k}=\left(h_{1, k}\left(y_{1}, y_{2}\right), h_{2, k}\left(y_{1}, y_{2}\right), 0\right)$.

Consider $P=P\left(x_{1}^{*}, x_{2}^{*}, x_{3}^{*}\right) \in \partial E$. If $P \in \partial E \backslash \partial E_{k}$, then $\left(\mathbf{a}-\mathbf{b}_{1}\right)_{k}(P)=0$ and we can take $\left(\mathbf{w}_{2}\right)_{k}(P)=\nabla\left(\mathbf{w}_{2}\right)_{k}(P)=0$.

On the other hand, if $P \in \partial E_{k}$, then the corresponding point

$$
Q=Q\left(y_{1}^{*}, y_{2}^{*}, 0\right) \in \operatorname{supp} \mathbf{h}_{k}
$$

and we can proceed with the construction of $\left(\mathbf{w}_{2}\right)_{k}$ as we have done before for the set $E=\left\{y_{3}>0\right\}$. The vector field $\left(\mathbf{w}_{2}\right)_{k}=\left(\mathbf{w}_{2}\right)_{k}\left(y_{1}, y_{2}, y_{3}\right)$ vanishes as $\left(y_{1}, y_{2}\right) \notin D_{k}$, but there is no condition on $y_{3}$. On the other hand, a careful inspection of the construction for $E=\left\{y_{3}>0\right\}$ shows that if we consider a function $f \in C_{o}^{\infty}(\mathbb{R})$ with $f(0)=0, f^{\prime}(0)=1, \operatorname{supp} f=[-r, r]$ and $r>0$ arbitrary, also the vector field

$$
\begin{aligned}
& w_{2,1}\left(x_{1}, x_{2}, x_{3}\right)=f\left(x_{3}\right) h_{2}\left(x_{1}, x_{2}\right) \\
& w_{2,2}\left(x_{1}, x_{2}, x_{3}\right)=-f\left(x_{3}\right) h_{1}\left(x_{1}, x_{2}\right) \\
& w_{2,3}\left(x_{1}, x_{2}, x_{3}\right)=0
\end{aligned}
$$

is a solution. Therefore, the support of $\left(\mathbf{w}_{2}\right)_{k}(y)$ can be contained in a neighborhood of $D_{k}$ of height $r$. Once $\left(\mathbf{w}_{2}\right)_{k}(y)$ has been built, applying the inverse change of variable, we obtain $\left(\mathbf{b}_{2}\right)_{k}(x)=\operatorname{curl}\left(\mathbf{w}_{2}\right)_{k}(x)$.

Since we have defined $\left(\mathbf{w}_{2}\right)_{k}(P)$ in two different ways, namely taking into account whether $P \in \partial E \backslash \partial E_{k}$ or $P \in \partial E_{k}$, we still need to check that the values of $\left(\mathbf{w}_{2}\right)_{k}$ and its derivatives are all compatible. Again, a careful control of the proof, shows that the only requirement is that the tangential derivatives have to vanish, and this is surely satisfied by our construction.

As for the smoothness of $\left(\mathbf{w}_{2}\right)_{k}$, it is a direct consequence of the smoothness of $\mathbf{a}$, and also of the smoothness of $\partial E$, which affects the regularity of the change of variables. In particular, if $\partial E$ is of class $C^{1}$ and has the segment property, and $\mathbf{a} \in W^{\frac{1}{2}, 2}(\partial E)$, then it is a matter of straightforward computations to see that the previous construction yields $\mathbf{b} \in W^{1,2}(E)$ and $\mathbf{b}=\operatorname{curl} \mathbf{w}_{1}+\operatorname{curl} \mathbf{w}_{2}$ with $\mathbf{w}_{1}, \mathbf{w}_{2} \in W^{2,2}(E)$. As for (4.9c), it is a consequence of standard elliptic estimates.

On the other hand, if we consider regularity in the class of continuous functions, once more it is relatively easy to see, as pointed out in [23], that if $\partial E$ is a $C^{2}$ surface and $\mathbf{a}$ is continuous on $\partial E$, then $\mathbf{b}$ is continuous on $\bar{E}$.

When $E \subset \mathbb{R}^{2}$, things are much easier. Recalling that $E$ is simply connected, we look for $\mathbf{b}$ in the form

$$
\mathbf{b}=\left(\frac{\partial w}{\partial x_{2}},-\frac{\partial w}{\partial x_{1}}\right)
$$

The condition $\left.\mathbf{b}\right|_{\partial E}=\mathbf{a}$ gives the values of $\frac{\partial w}{\partial n}$ and $\frac{\partial w}{\partial \tau}$ on $\partial E$. From the values of $\frac{\partial w}{\partial \tau}$ on $\partial E$, we determine $w$ on $\partial E$ up to an arbitrary constant, and $w$ is a single valued continuous function, as

$$
\int_{\partial E} \frac{\partial w}{\partial n} d \sigma=\int_{\partial E} \mathbf{a} \cdot \mathbf{n} d \sigma=0
$$

Once we know $\left.w\right|_{\partial E}$ and $\left.\frac{\partial w}{\partial n}\right|_{\partial E}$, we can finally build $w$ in $E$.

## 4.3c Proof of Proposition 4.3c

By Lemma 4.3c, if $N=3$ there exists $\mathbf{w} \in W^{2,2}(E)$ (if $N=2$, we have $\left.w \in W^{2,2}(E)\right)$ such that curl $\mathbf{w}(x)=\mathbf{a}(x)$ for any $x \in \partial E$. For any $\epsilon>0$, let $\varphi_{\epsilon} \in C^{\infty}(\bar{E})$ be the function built in Lemma 4.1c and set

$$
\boldsymbol{\psi} \stackrel{\text { def }}{=} \varphi_{\epsilon} \mathbf{w}, \quad \mathbf{b}=\operatorname{curl} \boldsymbol{\psi}=\operatorname{curl}\left(\varphi_{\epsilon} \mathbf{w}\right)
$$

By construction $\boldsymbol{\psi} \in W^{2,2}(E)$, it is compactly supported about $\partial E$, and for any $x \in \partial E$ we have

$$
\boldsymbol{\psi}(x)=\mathbf{w}(x) \quad \Rightarrow \quad \operatorname{curl} \boldsymbol{\psi}(x)=\operatorname{curl} \mathbf{w}(x)=\mathbf{a}(x) .
$$

Moreover, due to its very definition, and to (4.9c)

$$
\|\operatorname{curl} \boldsymbol{\psi}\|_{W^{1,2}(E)} \leq c\|\mathbf{a}\|_{W^{1 / 2,2}(\partial E)}
$$

It remains to show the validity of (4.7c). Lemma 4.1c yields

$$
|\mathbf{b}(x)| \leq \frac{\epsilon \kappa_{2}}{\delta(x)}|\mathbf{w}(x)|+|\nabla \mathbf{w}(x)| \quad \text { if } \quad \delta(x)<2 \gamma(\epsilon)
$$

and

$$
\mathbf{b}(x)=0 \quad \text { if } \delta(x) \geq 2 \gamma(\epsilon)
$$

By the Sobolev embedding Theorem

$$
\begin{aligned}
|\mathbf{w}(x)| & \leq c\|\mathbf{w}\|_{2,2} \\
\|\nabla \mathbf{w}\|_{3} & \leq c\|\mathbf{w}\|_{2,2}
\end{aligned}
$$

and by (4.9c) this implies

$$
\begin{equation*}
|\mathbf{w}(x)|+\|\nabla \mathbf{w}\|_{3} \leq c\|\mathbf{a}\|_{W^{1 / 2,2}(\partial E)} . \tag{4.12c}
\end{equation*}
$$

Therefore, for every $\mathbf{u} \in V$, we have

$$
\begin{aligned}
\left\|\left|\mathbf{u}\|\operatorname{curl} \boldsymbol{\psi} \mid\|_{2} \leq\right.\right. & c \epsilon\|\mathbf{a}\|_{W^{1 / 2,2}(\partial E)}\left\|\frac{1}{\delta} \mathbf{u}\right\|_{2} \\
& +\left(\int_{\delta(x)<2 \gamma(\epsilon)}|\mathbf{u}|^{2}|\nabla \mathbf{w}|^{2} d x\right)^{\frac{1}{2}} \\
\leq & c \epsilon\|\mathbf{a}\|_{W^{1 / 2,2}(\partial E)}\left\|\frac{1}{\delta} \mathbf{u}\right\|_{2}+\|\nabla \mathbf{u}\|_{2}\|\nabla \mathbf{w}\|_{3, E_{\epsilon}},
\end{aligned}
$$

where

$$
E_{\epsilon} \stackrel{\text { def }}{=}\{x \in E: \delta(x)<2 \gamma(\epsilon)\}
$$

Due to (4.12c), we have that $\zeta(\epsilon) \stackrel{\text { def }}{=}\|\nabla \mathbf{w}\|_{3 ; E_{\epsilon}}$ vanishes as $\epsilon \rightarrow 0$, whereas Lemma 4.2c yields $\left\|\delta^{-1} \mathbf{u}\right\|_{2} \leq c\|\nabla \mathbf{u}\|_{2}$; therefore, we can conclude

$$
\|\mathbf{u}\| \operatorname{curl} \boldsymbol{\psi} \mid\left\|_{2} \leq c\left(\epsilon\|\mathbf{a}\|_{W^{1 / 2,2}(\partial E)}+\zeta(\epsilon)\right)\right\| \nabla \mathbf{u}\left\|_{2}=\chi(\epsilon)\right\| \nabla \mathbf{u} \|_{2}
$$

where $\chi(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$.

## 4.4c The Case of a General Domain $E$

We briefly discuss what happens, when $E \subset \mathbb{R}^{N}, N=2$, 3 , is not simply connected, and/or the boundary $\partial E$ has multiple connected components $\Gamma_{i}$, $i=1, \ldots, m$ with $m>1$.

Consequently, the natural compatibility condition on the velocity $\mathbf{v}$ at the boundary $\partial E$, required by the incompressibility of the fluid, is

$$
\int_{\partial E} \mathbf{a} \cdot \mathbf{n} d \sigma=\sum_{i=1}^{m} \int_{\Gamma_{i}} \mathbf{a} \cdot \mathbf{n} d \sigma=0
$$

where $\mathbf{n}$ is the outward unit normal to $\partial E$, whereas the argument we have presented above (which is Leray's original argument in [26]) works if the condition

$$
\forall i=1, \ldots, m \quad \int_{\Gamma_{i}} \mathbf{a} \cdot \mathbf{n} d \sigma=0
$$

holds, which is obviously stronger. Moreover, such a stricter requirement does not allow for the presence of extended sinks and sources into the region of flow, which is physically interesting. The question of whether the problem we have considered here, admits a solution only under the natural restriction is a fundamental question in the mathematical theory of the Navier-Stokes equations.

We refrain from further elaborating on this issue here. The reader interested in the solenoidal extension to a bounded open set which is not simply connected, and/or has a boundary with multiple connected components, can refer, for example, to [13], Section 2 and to [16], Chapter VIII.

## 5c Recovering the Pressure

## 5.1c Proof of Proposition 5.1 for $u \in H^{\perp} \cap C^{\infty}\left(E ; \mathbb{R}^{3}\right)$

Pick $\mathbf{w} \in C_{o}^{\infty}\left(E ; \mathbb{R}^{3}\right)$. Then curl $\mathbf{w} \in \mathcal{V}$ and hence, by the membership $\mathbf{u} \in$ $H^{\perp}$, and by integration by parts

$$
\int_{E} \mathbf{u} \cdot \operatorname{curl} \mathbf{w} d x=-\int_{E} \operatorname{curl} \mathbf{u} \cdot \mathbf{w} d x=0 \quad \text { for all } \mathbf{w} \in C_{o}^{\infty}\left(E ; \mathbb{R}^{3}\right)
$$

By density this continues to hold for all $\mathbf{w} \in L^{2}\left(E ; \mathbb{R}^{3}\right)$. Therefore if $\mathbf{u} \in$ $H^{\perp} \cap C_{o}^{\infty}\left(E ; \mathbb{R}^{3}\right)$ then curl $\mathbf{u}=0$ in $E$. Since $E$ is assumed to be convex, denoting by $\boldsymbol{\eta}$ the coordinates in $\mathbb{R}^{3}$, the latter is a necessary and sufficient condition for the differential form $d \mathbf{u}=\mathbf{u} \cdot d \boldsymbol{\eta}$ to be exact in $E$. Having fixed $x, y \in E$ consider a smooth path from $y$ to $x$, i.e.,

$$
\gamma_{x, y}=\left\{\boldsymbol{\eta} \in C^{1}\left[(\alpha, \beta) ; \mathbb{R}^{3}\right], \boldsymbol{\eta}(\alpha)=y, \boldsymbol{\eta}(\beta)=x ;\left|\boldsymbol{\eta}^{\prime}\right|>0 .\right\}
$$

The path integral

$$
\begin{equation*}
p(x, y)=\int_{\gamma_{x, y}} d \mathbf{u}=\int_{\alpha}^{\beta} \mathbf{u}(\boldsymbol{\eta}(s)) \cdot \boldsymbol{\eta}^{\prime}(s) d s \tag{5.1c}
\end{equation*}
$$

is independent of $\gamma_{x, y}$, and, for a fixed $y \in E$, uniquely defines a function $p(\cdot, y)$ satisfying $\nabla p(\cdot, y)=\mathbf{u}$. Moreover, for any $y_{1}, y_{2} \in E$, by the stated independence of the path integral, $p\left(\cdot, y_{2}\right)=p\left(\cdot, y_{1}\right)+p\left(y_{1}, y_{2}\right)$. Since $\mathbf{u} \in$ $C_{o}^{\infty}\left(E ; \mathbb{R}^{3}\right)$ one has $p(\cdot, y) \in C^{\infty}(E)$. To establish the proposition in the case $\mathbf{u} \in H^{\perp} \cap C^{\infty}\left(E ; \mathbb{R}^{3}\right)$, fix $y \in E$ and determine the function $E \ni x \rightarrow p(x, y)$ up to a constant.

## 5.2c Proof of Proposition 5.1 for $\mathbf{u} \in H^{\perp}$

Having fixed $\mathbf{u} \in H^{\perp}$, regard it as defined in $\mathbb{R}^{3}$ by extending it to zero outside $E$. Pick $\mathbf{w} \in C_{o}^{\infty}(E)$, and for $\epsilon>0$ denote by $\mathbf{w}_{\epsilon}=J_{\epsilon} * \mathbf{w}$ the $\epsilon$-mollification of $\mathbf{w}$ by the Friedrich's mollifying kernel $J_{\epsilon}(\cdot)$. We choose $\epsilon$ sufficiently small, such that $\mathbf{w}_{\epsilon} \in C_{o}^{\infty}\left(E ; \mathbb{R}^{3}\right)$ and curl $\mathbf{w}_{\epsilon}$ restricted to $E$ is in $H$. Since $\mathbf{u} \in H^{\perp}$

$$
\begin{aligned}
0 & =\int_{E} \mathbf{u} \cdot \operatorname{curl} \mathbf{w}_{\epsilon} d x=\int_{\mathbb{R}^{3}} \mathbf{u} \cdot \operatorname{curl} \mathbf{w}_{\epsilon} d x \\
& =\int_{\mathbb{R}^{3}} \mathbf{u}_{\epsilon} \cdot \operatorname{curl} \mathbf{w} d x=\int_{E} \mathbf{u}_{\epsilon} \cdot \operatorname{curl} \mathbf{w} d x \\
& =-\int_{E} \operatorname{curl} \mathbf{u}_{\epsilon} \cdot \mathbf{w} d x=0
\end{aligned}
$$

for all $\mathbf{w} \in C_{o}^{\infty}\left(E ; \mathbb{R}^{3}\right)$. By density this continues to hold for all $\mathbf{w} \in L^{2}\left(E ; \mathbb{R}^{3}\right)$. Therefore, curl $\mathbf{u}_{\epsilon}=0$ in $E$, and the path integral

$$
p(x, y ; \epsilon)=\int_{\gamma_{x, y}} d \mathbf{u}_{\epsilon}=\int_{\alpha}^{\beta} \mathbf{u}_{\epsilon}(\boldsymbol{\eta}(s)) \cdot \boldsymbol{\eta}^{\prime}(s) d s
$$

is independent of $\gamma_{x, y} \subset E$. For a fixed $y \in E$, such an integral uniquely defines a function $p(\cdot, y ; \epsilon)$ satisfying $\nabla p(\cdot, y ; \epsilon)=\mathbf{u}_{\epsilon}$. Moreover, for any $y_{1}, y_{2} \in$ $E$, by the stated independence of the path integral, $p\left(\cdot, y_{2} ; \epsilon\right)=p\left(\cdot, y_{1} ; \epsilon\right)+$ $p\left(y_{1}, y_{2} ; \epsilon\right)$.

Proposition 5.1c There exists $p(\cdot, \cdot) \in L^{2}(E \times E)$ and a subnet $\left\{p\left(\cdot, \cdot ; \epsilon^{\prime}\right)\right\} \subset$ $\{p(\cdot, \cdot ; \epsilon)\}$, relabeled with $\epsilon$ such that as $\epsilon \rightarrow 0$

$$
\begin{array}{ll}
p(\cdot, \cdot ; \epsilon) \rightarrow p(\cdot, \cdot) & \text { in } L^{2}(E \times E) \text { and a.e. in } E \times E \\
p(\cdot, y ; \epsilon) \rightarrow p(\cdot, y) & \text { in } L^{2}(E) \text { for a.e. } y \in E \\
p\left(x, y_{2}\right)=p\left(x, y_{1}\right)+p\left(y_{1}, y_{2}\right) & \text { a.e. in } E \times E  \tag{5.2c}\\
\nabla p(\cdot, y ; \epsilon) \rightarrow \nabla p(\cdot, y) & \text { weakly in } L^{2}(E) \text { for a.e. } y \in E \\
\nabla p(\cdot, y)=\mathbf{u} & \text { for a.e. } y \in E .
\end{array}
$$

We rely on the following result.
Lemma 5.1c There holds:

$$
\|p(\cdot, \cdot ; \epsilon)\|_{2 ; E \times E} \leq 2 \sqrt{2 \pi} \operatorname{diam}(E)^{\frac{3}{2}}\left\|\mathbf{u}_{\epsilon}\right\|_{2 ; E} \leq 2 \sqrt{2 \pi} \operatorname{diam}(E)^{\frac{3}{2}}\|\mathbf{u}\|_{2 ; E}
$$

uniformly in $\epsilon$.
Proof. Fix $\epsilon>0$ and in computing $p(x, y ; \epsilon)$ from (5.1c) take the segment

$$
(0,|x-y|) \ni s \rightarrow y+s \nu \quad \text { where } \quad \nu=\frac{x-y}{|x-y|}
$$

For such a choice, and Hölder's inequality

$$
p^{2}(x, y ; \epsilon) \leq \operatorname{diam}(E) \int_{0}^{|x-y|}\left|\mathbf{u}_{\epsilon}\right|^{2}(y+s \nu) d s
$$

Integrate both sides in $d x$ over $E$, and compute the resulting integral on the right-hand side in polar coordinates with pole at $y$ and angular variable $\nu$ ranging over the unit sphere of $\mathbb{R}^{3}$. Denote by $R(y, \nu)$ the polar representation of $\partial E$ with pole at $y$ and set also $z=y+s \nu$ so that the polar radius is $s=|z-y|$ and ranges over $(0, R(y, \nu))$. This gives

$$
\begin{aligned}
\|p(\cdot, y ; \epsilon)\|_{2 ; E}^{2} & \leq \operatorname{diam}(E)^{2} \int_{|\nu|=1}\left(\int_{0}^{R(y, \nu)}\left|\mathbf{u}_{\epsilon}\right|^{2}(y+s \nu) d s\right) d \nu \\
& =\operatorname{diam}(E)^{2} \int_{\|\nu\|=1}\left(\int_{0}^{R(y, \nu)} \frac{\left|\mathbf{u}_{\epsilon}(z)\right|^{2}}{|z-y|^{2}}|z-y|^{2} d|z-y|\right) d \nu \\
& =\operatorname{diam}(E)^{2} \int_{E} \frac{\left|\mathbf{u}_{\epsilon}(z)\right|^{2}}{|z-y|^{2}} d z
\end{aligned}
$$

Next, integrate both sides in $d y$ over $E$ and estimate the resulting integral on the right-hand side by making use of Fubini's theorem to obtain

$$
\begin{aligned}
\|p(\cdot, \cdot ; \epsilon)\|_{2 ; E \times E}^{2} & \leq \operatorname{diam}(E)^{2} \int_{E}\left|\mathbf{u}_{\epsilon}(z)\right|^{2} d z \sup _{z \in E} \int_{E} \frac{1}{|z-y|^{2}} d y \\
& \leq 8 \pi \operatorname{diam}(E)^{3} \int_{E}\left|\mathbf{u}_{\epsilon}(z)\right|^{2} d z \\
& \leq 8 \pi \operatorname{diam}(E)^{3}\|\mathbf{u}\|_{2 ; E}^{2}
\end{aligned}
$$

The last inequality follows from the properties of the mollifying kernels.
Corollary 5.1c For all positive $\epsilon_{1}, \epsilon_{2}$

$$
\left\|p\left(\cdot, \cdot ; \epsilon_{1}\right)-p\left(\cdot, \cdot ; \epsilon_{2}\right)\right\|_{2 ; E \times E} \leq 2 \sqrt{2 \pi} \operatorname{diam}(E)^{\frac{3}{2}}\left\|\mathbf{u}_{\epsilon_{1}}-\mathbf{u}_{\epsilon_{2}}\right\|_{2 ; E}
$$

Proof (of Proposition 5.1c). Since $\left\{\mathbf{u}_{\epsilon}\right\}$ is Cauchy in $L^{2}\left(E ; \mathbb{R}^{3}\right)$ the net $\{p(\cdot, \cdot ; \epsilon)\}$ is Cauchy in $L^{2}(E \times E)$ and by the completeness of $L^{2}(E \times E)$ there exists $p(\cdot, \cdot) \in L^{2}(E \times E)$ such that

$$
\lim _{\epsilon \rightarrow 0}\|p(\cdot, \cdot ; \epsilon)-p(\cdot, \cdot)\|_{2 ; E \times E}=0
$$

Subnets can now be selected satisfying the first three statements in (5.2c). Fix $y \in E$ for which the second of (5.2c) holds and for $\zeta \in C_{o}^{\infty}\left(E ; \mathbb{R}^{3}\right)$ compute

$$
\begin{gathered}
\lim _{\epsilon \rightarrow 0}\langle\nabla p(\cdot, y ; \epsilon), \boldsymbol{\zeta}\rangle_{L^{2}(E)}=\lim _{\epsilon \rightarrow 0}-\langle p(\cdot, y ; \epsilon), \operatorname{div} \boldsymbol{\zeta}\rangle_{L^{2}(E)} \\
=\langle\mathbf{u}, \boldsymbol{\zeta}\rangle_{L^{2}(E)}=-\langle p(\cdot, y), \operatorname{div} \boldsymbol{\zeta}\rangle_{L^{2}(E)}
\end{gathered}
$$

## 5.3c More General Versions of Proposition 5.1

5.1. Convexity of $E$ has been used in the previous proof, in order to conclude that $d \mathbf{u}$ is exact. Prove that Proposition 5.1c continues to hold if $E$ is not convex, but any two points $x, y \in E$ can be connected by a smooth curve $\gamma_{x, y} \subset E$ of length not exceeding a fixed constant $L$. This would include bounded, simply connected sets $E$ with smooth boundary $\partial E$.
5.2. If $E$ is unbounded let $E_{n}=E \cap\{|x|<n\}$ and assume that each $E_{n}$ satisfies the condition in $\mathbf{5 . 1}$ with the constant $L_{n}$ possibly depending on $n$. State and prove a local version of Proposition 5.1.
5.3. The Helmholtz-Weyl decomposition, sometimes also referred to as Hodge decomposition, can be actually proven for any open set $E \subset \mathbb{R}^{N}$, if one works in $L^{2}\left(E ; \mathbb{R}^{N}\right)$, as it is the case here (see [15], § III.1). The situation is more complicated if one works in $L^{p}\left(E ; \mathbb{R}^{N}\right)$ with $p \in(1, \infty)$, $p \neq 2$.

## 8c Time-Dependent Navier-Stokes Equations in Bounded Domains

In Section 8 we considered the Navier-Stokes equations in $E_{T}$ with $E \subset \mathbb{R}^{3}$ an open, bounded set with smooth boundary, and stated Hopf's 1951 result about the existence of weak solutions of the initial-boundary value problem (8.1) ([20]).

As a matter of fact, the first result about the existence of weak solutions dates back to 1934 and is due to Leray ([27]), who studied the problem in the whole space $\mathbb{R}^{3}$ with divergence free initial condition $u_{o} \in L^{2}\left(\mathbb{R}^{3}\right)$. Somehow, the more difficult case was solved first.

## 10c Selecting Subsequences Strongly Convergent in $L^{2}\left(E_{T}\right)$

Lemma 10.1c (Friedrichs [14]) Let $Q \subset \mathbb{R}^{N}$ be a cube of edge $L$ and let $\mathbf{u} \in W^{1, p}(Q)$ for some $1<p<N$. For every $\varepsilon>0$ there exist a positive integer $k_{\varepsilon}$ depending only on $\varepsilon$ and $L$, and independent of $\mathbf{u}$, and $k_{\varepsilon}$ linearly independent functions $\left\{\boldsymbol{\psi}_{\ell}\right\}_{\ell=1}^{k_{\varepsilon}} \subset L^{p}(Q)$ such that

$$
\begin{equation*}
\|\mathbf{u}\|_{p ; Q}^{p} \leq \sum_{\ell=1}^{k_{\varepsilon}}\left|\int_{Q} \mathbf{u} \cdot \boldsymbol{\psi}_{\ell} d x\right|^{p}+\varepsilon\|\nabla \mathbf{u}\|_{p ; Q}^{p} \tag{10.1c}
\end{equation*}
$$

Remark 10.1c The conclusion continues to hold if $u \in W_{o}^{1, p}(E)$, where $E$ is a bounded open set in $\mathbb{R}^{N}$. Indeed $E$ can be included in a cube $Q$ and, since $u$ has zero trace on $\partial E$, it can be extended in the whole cube by setting it to be zero outside $E$.

## 10.1c Proof of Friedrichs Lemma

The starting point is Poincaré inequality which we state next. Let

$$
u_{Q}=\frac{1}{|Q|} \int_{Q} u d x=f_{Q} u d x
$$

denote the integral average of $u$ over $Q$.
Theorem 10.1c (Poincaré Inequality). Let $u \in W^{1, p}(Q)$. There exists a constant $\gamma$ depending only on the dimension $N$ and $p$, such that

$$
\begin{equation*}
\left\|u-u_{Q}\right\|_{p_{*} ; Q} \leq \gamma\|\nabla u\|_{p ; Q} \quad \text { where } \quad p_{*}=\frac{N p}{N-p} \tag{10.2c}
\end{equation*}
$$

Proof. See [6], Chapter 10, § 10.1.

Corollary 10.1c Let $u \in W^{1, p}(Q)$. There exists a constant $\gamma$ depending only on the dimension $N$ and $p$, such that

$$
\begin{equation*}
\left\|u-u_{Q}\right\|_{p ; Q} \leq \gamma L\|\nabla u\|_{p ; Q} \tag{10.3c}
\end{equation*}
$$

Proof. Apply Hölder's inequality to $\left\|u-u_{Q}\right\|_{p ; Q}$ and use (10.2c).
Let $k$ be a positive integer to be chosen and subdivide $Q$ in $k^{N}$ equal subcubes $Q_{\ell}$, with pairwise disjoint interior and edge $L / k$. Then compute and estimate

$$
\begin{aligned}
\int_{Q}|u|^{p} d x & =\sum_{\ell=1}^{k^{N}} \int_{Q_{\ell}}|u|^{p} d x=\sum_{\ell=1}^{k^{N}} \int_{Q_{\ell}}\left|\left(u-u_{Q_{\ell}}\right)+u_{Q_{\ell}}\right|^{p} d x \\
& \leq 2^{p-1}\left(\frac{k}{L}\right)^{N(p-1)} \sum_{\ell=1}^{k^{N}}\left|\int_{Q_{\ell}} u d x\right|^{p}+2^{p-1} \sum_{\ell=1}^{k^{N}} \int_{Q_{\ell}}\left|u-u_{Q_{\ell}}\right|^{p} d x \\
& \leq \sum_{\ell=1}^{k^{N}}\left|\int_{Q} u \psi_{\ell} d x\right|^{p}+\gamma 2^{p-1} \frac{L}{k} \sum_{\ell=1}^{k^{N}} \int_{Q_{\ell}}|\nabla u|^{p} d x \\
& =\sum_{\ell=1}^{k^{N}}\left|\int_{Q} u \psi_{\ell} d x\right|^{p}+\varepsilon \int_{Q}|\nabla u|^{p} d x
\end{aligned}
$$

where we have set

$$
\psi_{\ell}=2^{\frac{p-1}{p}}\left(\frac{k}{L}\right)^{N \frac{p-1}{p}} \chi_{Q_{\ell}} \quad \text { and } \quad \varepsilon=\gamma 2^{p-1} \frac{L}{k}
$$

## 10.2c Compact Embedding of $W^{1, p}$ into $L^{q}(Q)$ for $1 \leq q<p_{*}$

- Prove a version of (10.3c) with the left-hand side replaced by $\left\|u-u_{Q}\right\|_{q ; Q}$ for $1 \leq q<p_{*}$.
- Prove a version of Friedrichs lemma with the left-hand side of (10.1c) replaced by $\|u\|_{q ; Q}$.
- Use such a version to prove the indicated compact embedding.

If $E$ is bounded, give conditions on $\partial E$ so that $u \in W^{1, p}(E)$ can be extended into a cube containing $E$ with $u \in W^{1, p}(Q)$.

## 10.3c Solutions Global in Time

Let $\mathbf{f} \in L^{2}\left(E_{T} ; \mathbb{R}^{3}\right)$. Prove that a Hopf solution of (8.1) satisfies

$$
\begin{align*}
& \|\mathbf{v}(t)\|_{2 ; E} \leq\left\|\mathbf{v}_{o}\right\|_{2 ; E}+\|\mathbf{f}\|_{2 ; E_{t}} \\
& \|\nabla \mathbf{v}\|_{2 ; E_{t}} \leq \frac{1}{\nu \sqrt{2}}\left(\left\|\mathbf{v}_{o}\right\|_{2 ; E}+\|\mathbf{f}\|_{2 ; E_{t}}\right) \quad \text { for a.e. } t \in(0, T) . \tag{10.4c}
\end{align*}
$$

If $\mathbf{f} \in L^{2}\left(\mathbb{R}^{+} ; L^{2}\left(E ; \mathbb{R}^{3}\right)\right)$, then (8.1) has a weak solution global in time, i.e., in $E \times \mathbb{R}^{+}$. Moreover, such a solution satisfies the energy estimates (10.4c) for all $t \in \mathbb{R}^{+}$.

## 11c The Limiting Process and Proof of Theorem 8.1

In Section 11 we underlined that the strong convergence is needed to pass to the limit in the non-linear term. We now discuss a counterexample, in order to show that weak convergence in general does not suffice.

In particular, we consider a sequence $\left\{\mathbf{v}_{n}(x, t)\right\} \subset L^{2}\left(0, T ; W^{1,2}(E)\right) \cap$ $L^{\infty}\left(0, T ; L^{2}(E)\right)$ which satisfies the Navier-Stokes equations (8.1) with $\mathbf{f}=0$ in the weak sense of (8.2). Moreover, we assume that
a) $x \mapsto \mathbf{v}_{n}(x, t) \in C^{\infty}(E)$ for a.e. $t$, uniformly in $n$;
b) $\frac{\partial^{h} \mathbf{v}_{n}}{\partial x_{k}^{h}}(x, t) \in L^{\infty}\left(E_{T}\right), k=1,2, \ldots, N, h \in \mathbb{N}$, uniformly in $n$;
c) $\frac{\partial}{\partial t} \Delta \mathbf{v}_{n} \in L^{\infty}\left(E_{T}\right)$, uniformly in $n$.

In spite of the great regularity of a)-b)-c), we show that we do not have

$$
\int_{0}^{T} \int_{E}\left(\mathbf{v}_{n} \cdot \nabla\right) \mathbf{v}_{n} \cdot \varphi d x d t \rightarrow \int_{0}^{T} \int_{E}(\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \varphi d x d t
$$

where $\mathbf{v}$ is the weak limit of $\mathbf{v}_{n}$ in $L^{2}\left(E_{T}\right)$, and $\varphi \in C_{o}^{\infty}\left(E_{T} ; \mathbb{R}^{N}\right)$.
The counterexample is built in the following way. Let $\psi$ be harmonic in $E$, i.e. it satisfies $\operatorname{div} \nabla \psi \equiv \Delta \psi=0$. Set

$$
\mathbf{v}_{n}(x, t)=a_{n}(t) \nabla \psi
$$

where

$$
\left\{a_{n}\right\} \subset L^{\infty}(0, T) \text { uniformly in } n, \text { and }\left\{a_{n}\right\} \subset C^{1}(0, T)
$$

Then $\mathbf{v}_{n}$ satisfies (a)-(b)-(c) above. Moreover, $\mathbf{v}_{n}$ satisfies the Navier-Stokes equations, that is

$$
\begin{aligned}
\int_{0}^{T} \int_{E} a_{n}^{\prime}(t) \nabla \psi \cdot \varphi & d x d t-\int_{0}^{T} a_{n}(t) \int_{E} \Delta(\nabla \psi) \cdot \varphi d x d t \\
& +\int_{0}^{T} a_{n}^{2}(t) \int_{\Omega} \frac{\partial \psi}{\partial x_{i}} \frac{\partial^{2} \psi}{\partial x_{i} \partial x_{j}} \varphi d x d t=I_{1}+I_{2}+I_{3}=0
\end{aligned}
$$

for every $\boldsymbol{\varphi} \in C^{\infty}(0, T ; \mathcal{V})$. Indeed, we have $I_{1}=I_{2}=0$ trivially (we rely on the integration by parts in $I_{1}$ ). For $I_{3}$ we have

$$
\begin{aligned}
\int_{E} \frac{\partial \psi}{\partial x_{i}} \frac{\partial^{2} \psi}{\partial x_{i} \partial x_{j}} \boldsymbol{\varphi} d x & =\frac{1}{2} \int_{E} \frac{\partial}{\partial x_{j}}\left(\sum_{i=1}^{N}\left(\frac{\partial \psi}{\partial x_{i}}\right)^{2}\right) \boldsymbol{\varphi} d x \\
& =-\frac{1}{2} \int_{E}|\nabla \psi|^{2} \operatorname{div} \varphi d x=0
\end{aligned}
$$

Consider now a sequence $\left\{a_{n}\right\} \subset L^{\infty}(0, T)$ such that $\left\|a_{n}(t)\right\|_{L^{2}(0, T)}=1$ and $a_{n}(t) \rightharpoonup 0$ weakly in $L^{2}(0, T)$; for example, we could take $a_{n}(t)=\sqrt{\frac{2}{T}} \sin \frac{n \pi t}{T}$. Then

$$
\mathbf{v}_{n}(x, t)=a_{n}(t) \nabla \psi \rightharpoonup 0
$$

weakly in $L^{2}\left(E_{T}\right)$, but if we consider a general $\varphi \in C_{o}^{\infty}\left(E_{T} ; \mathbb{R}^{N}\right)$

$$
\begin{aligned}
\int_{0}^{T} \int_{E}\left(\mathbf{v}_{n} \cdot \nabla\right) \mathbf{v}_{n} \cdot \varphi d x d t & =\frac{1}{2} \int_{0}^{T} a_{n}^{2}(t)\left[\int_{E} \nabla\left(|\nabla \psi|^{2}\right) \cdot \varphi d x\right] d t \\
& =-\frac{1}{2}\left\|a_{n}\right\|_{L^{2}(0, T)}^{2}\left(\int_{E}|\nabla \psi|^{2} \operatorname{div} \varphi d x\right) \\
& =-\frac{1}{2}\left(\int_{E}|\nabla \psi|^{2} \operatorname{div} \varphi d x\right) \neq 0
\end{aligned}
$$

## 12c Higher Integrability and Some Consequences

12.1. Explain why (8.2) holding for all $\boldsymbol{\varphi} \in C^{\infty}(0, T ; \mathcal{V})$ implies (12.2) holding weakly for all $\varphi \in C_{o}^{\infty}\left(E_{T} ; \mathbb{R}^{N}\right)$.

## 13c Energy Identity for the Homogeneous Boundary Value Problem with Higher Integrability

The proof of Proposition 13.1 essentially gives a way of taking $\varphi=\mathbf{v}$ in the weak formulation (8.2).

Proposition 13.1c Let $\mathbf{v}$ be a weak solution of (8.2) ${ }_{0}$. Assume moreover that $\mathbf{v} \in L^{p, q}\left(E_{T} ; \mathbb{R}^{N}\right)$ with $p>N$ and $q>2$ satisfying (12.3). Then $\mathbf{v}$ satisfies the energy estimates (10.4c).
By the same token, Proposition 14.1 can be extended several ways. For example, one may permit $\mathbf{f}$ not to be zero, or the boundary data for $\mathbf{v}$ and $\mathbf{u}$ not to be zero, provided $\mathbf{w}=(\mathbf{v}-\mathbf{u})$ has zero trace on $\partial E$. State and prove version of such facts by writing the corresponding weak formulation for $\mathbf{w}$ and taking $\varphi=\mathbf{w}$ in the indicated approximate sense. This is possible by the assumed higher integrability on both $\mathbf{v}$ and $\mathbf{u}$ and hence $\mathbf{w}$. For $N=2$ such a higher integrability assumption is redundant.

## 15c Local Regularity of Solutions with Higher Integrability

The proofs of Theorem 15.1 in [40, 49] are based on a smart study of the vorticity equation (18.2). This is why the pressure does not appear in the statement. A careful analysis of the proof shows that the transport term is dealt with, as if it were an external force.

For a different approach see [38], and also the references therein. Moreover, in [38] Seregin extends his formulation of the regularity estimates up to the boundary under homogeneous Dirichlet conditions on a half cylinder.

## 16c Proof of Theorem 15.1- Introductory Results

Proposition 16.1c Let $k \in L^{p, p^{\prime}}\left(\mathbb{R}^{N} \times \mathbb{R}^{M} ; \mathbb{R}\right)$ and $g \in L^{q, q^{\prime}}\left(\mathbb{R}^{N} \times \mathbb{R}^{M} ; \mathbb{R}\right)$ with $N, M \geq 1$, and

$$
1 \leq q \leq r, \quad 1 \leq q^{\prime} \leq r^{\prime}, \quad \frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1, \quad \frac{1}{p^{\prime}}+\frac{1}{q^{\prime}}=\frac{1}{r^{\prime}}+1
$$

Then for the double convolution

$$
h(x, t) \stackrel{\text { def }}{=} \iint_{\mathbb{R}^{N} \times \mathbb{R}^{M}} k(x-\xi, t-\tau) g(\xi, \tau) d \xi d \tau
$$

we have

$$
\|h\|_{r, r^{\prime}} \leq\|k\|_{p, p^{\prime}}\|g\|_{q, q^{\prime}}
$$

Proof. First of all, consider the convolution only in one variable, namely

$$
(k * g)(x)=\int_{\mathbb{R}^{N}} k(x-\xi) g(\xi) d \xi
$$

where $k \in L^{p}\left(\mathbb{R}^{N}\right), g \in L^{q}\left(\mathbb{R}^{N}\right)$, and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1$. We have

$$
\begin{aligned}
|(k * g)(x)| & =\left|\int_{\mathbb{R}^{N}} k(x-\xi) g(\xi) d \xi\right| \\
& \leq \int_{\mathbb{R}^{N}}|k(x-y)| \cdot|g(\xi)| d \xi \\
& =\int_{\mathbb{R}^{N}}|k(x-\xi)|^{p / r}|g(\xi)|^{q / r}|k(x-\xi)|^{\frac{r-p}{r}}|g(\xi)|^{\frac{r-q}{r}} d \xi
\end{aligned}
$$

If we apply Hölder's inequality, we conclude that

$$
\begin{aligned}
|(k * g)(x)| \leq & {\left[\int_{\mathbb{R}^{N}}|k(x-\xi)|^{p}|g(\xi)|^{q} d \xi\right]^{1 / r} \cdot } \\
& \cdot\left[\int_{\mathbb{R}^{N}} \|\left. k(x-\xi)\right|^{p} d \xi\right]^{\frac{r-p}{r p}}\left[\int_{\mathbb{R}^{N}}|g(\xi)|^{q} d \xi\right]^{\frac{r-q}{r q}} \\
= & {\left[\int_{\mathbb{R}^{N}}|k(x-\xi)|^{p}|g(\xi)|^{q} d \xi\right]^{1 / r}\|k\|^{\frac{r-p}{r}}\|g\|^{\frac{r-q}{r}} }
\end{aligned}
$$

Raising both sides to the power $r$ yields

$$
|(k * g)(x)|^{r} \leq\left[\int_{\mathbb{R}^{N}}|k(x-\xi)|^{p}|g(\xi)|^{q} d \xi\right]\|k\|_{p}^{r-p}\|g\|_{q}^{r-q}
$$

If we now integrate with respect to $x$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^{N}}|(k * g)(x)|^{r} d x \leq & \int_{\mathbb{R}^{N}}|g(\xi)|^{q}\left[\int_{\mathbb{R}^{N}}|k(x-\xi)|^{p} d x\right] d \xi \\
& \cdot\|k\|_{p}^{r-p}\|g\|_{q}^{r-q}
\end{aligned}
$$

and we conclude that

$$
\begin{equation*}
\|k * g\|_{r} \leq\|k\|_{p}\|g\|_{q} . \tag{16.1c}
\end{equation*}
$$

The previous proof holds for any $N \geq 1$; inequality (16.1c) is usually known as Young's inequality for the convolution. Now, we want to consider the double convolution with respect to $(x, t) \in \mathbb{R}^{N} \times \mathbb{R}^{M}$, namely

$$
h(x, t)=\iint_{\mathbb{R}^{N} \times \mathbb{R}^{M}} k(x-\xi, t-\tau) g(\xi, \tau) d \xi d \tau
$$

We have

$$
\begin{aligned}
\|h(t)\|_{r} & =\left[\int_{\mathbb{R}^{N}}\left|\iint_{\mathbb{R}^{N} \times \mathbb{R}^{M}} k(x-\xi, t-\tau) g(\xi, \tau) d \xi d \tau\right|^{r} d x\right]^{1 / r} \\
& \leq\left[\int_{\mathbb{R}^{N}}\left[\int_{\mathbb{R}^{M}}\left|\int_{\mathbb{R}^{N}} k(x-\xi, t-\tau) g(\xi, \tau) d \xi\right| d \tau\right]^{r} d x\right]^{1 / r}
\end{aligned}
$$

For simplicity, let us set for the moment

$$
\int_{\mathbb{R}^{N}} k(x-\xi, t-\tau) g(\xi, \tau) d \xi=f(x, t, \tau)
$$

Then, we have

$$
\|h(t)\|_{r} \leq\left(\int_{\mathbb{R}^{N}}\left[\int_{\mathbb{R}^{M}}|f(x, t, \tau)| d \tau\right]^{r} d x\right)^{1 / r}
$$

We can apply the continuous Minkowski inequality (see, for example, [6, Chapter 6, Prop. 3.3]) to obtain

$$
\begin{aligned}
\|h(t)\|_{r} & \leq \int_{\mathbb{R}^{M}}\|f(x, t, \tau)\|_{r} d \tau \\
& =\int_{\mathbb{R}^{M}}\left\|\int_{\mathbb{R}^{N}} k(x-\xi, t-\tau) g(\xi, \tau) d \xi\right\|_{r} d \tau \\
& \leq \int_{\mathbb{R}^{M}}\|k(t-\tau)\|_{p}\|g(\tau)\|_{q} d \tau
\end{aligned}
$$

where we have taken (16.1c) into account. Let us momentarily set

$$
u(t-\tau)=\|k(t-\tau)\|_{p}, \quad v(\tau)=\|g(\tau)\|_{q}
$$

We can rewrite

$$
\|h(t)\|_{r} \leq \int_{\mathbb{R}^{M}} u(t-\tau) v(\tau) d \tau=(u * v)(t)
$$

Once more, by (16.1c) we conclude

$$
\begin{aligned}
\|h\|_{r, r^{\prime}} & =\left(\int_{\mathbb{R}^{M}}\|h(t)\|_{r}^{r^{\prime}} d t\right)^{1 / r^{\prime}}=\|u * v\|_{r^{\prime}} \leq\|u\|_{p^{\prime}}\|v\|_{q^{\prime}} \\
& =\left(\int_{\mathbb{R}^{M}}\|k(t-\tau)\|_{p}^{p^{\prime}} d t\right)^{1 / p^{\prime}}\left(\int_{\mathbb{R}^{M}}\|g(\tau)\|_{q}^{q^{\prime}} d \tau\right)^{1 / q^{\prime}}=\|k\|_{p, p^{\prime}}\|g\|_{q, q^{\prime}}
\end{aligned}
$$

Proposition 16.2c Let $k \in L^{p, p^{\prime}}\left(\mathbb{R}^{N} \times \mathbb{R} ; \mathbb{R}\right)$ and $g \in L^{q, q^{\prime}}\left(\Omega \times\left(t_{1}, t_{2}\right) ; \mathbb{R}\right)$ with $N \geq 1, \Omega$ a bounded domain in $\mathbb{R}^{N},\left(t_{1}, t_{2}\right) \subset(0, \infty)$, and

$$
1 \leq q \leq r, \quad 1 \leq q^{\prime} \leq r^{\prime}, \quad \frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1, \quad \frac{1}{p^{\prime}}+\frac{1}{q^{\prime}}=\frac{1}{r^{\prime}}+1
$$

Then for the double convolution

$$
h(x, t) \stackrel{\text { def }}{=} \iint_{\Omega \times\left(t_{1}, t_{2}\right)} k(x-\xi, t-\tau) g(\xi, \tau) d \xi d \tau, \quad(x, t) \in \Omega \times\left(t_{1}, t_{2}\right)
$$

we have

$$
\|h\|_{r, r^{\prime}} \leq\|k\|_{p, p^{\prime}}\|g\|_{q, q^{\prime}}
$$

Proof. Same as in Proposition 16.1c

## 20c Recovering the Pressure in the Time-Dependent Equations

In Section 20 we study the regularity of the pressure $p$ for weak solutions of (8.1) in $E_{T}$, where $E$ is a bounded, smooth domain of $\mathbb{R}^{3}$.

In the whole space $\mathbb{R}^{3}$ the situation is definitely simpler, and we sketch how the analogous corresponding result can be obtained. We follow an argument given in [3].

If we take the divergence of (8.1), we obtain

$$
\Delta p=-\sum_{i, j=1}^{3} \frac{\partial^{2}}{\partial x_{i} \partial x_{j}}\left(v_{i} v_{j}\right)
$$

in the sense of distributions in $\mathbb{R}^{3} \times(0, T)$, and therefore, in $\mathbb{R}^{3} \times\{t\}$ for a.e. $t \in(0, T)$.

Here $p$ is the sum of classical singular integral operators applied to $v_{i} v_{j}$. By the Calderón-Zygmund theory (see [47]), we have

$$
\int_{0}^{T} \int_{\mathbb{R}^{3}}|p|^{q} d x d t \leq C(q) \int_{0}^{T} \int_{\mathbb{R}^{3}}|\mathbf{v}|^{2 q}, \quad q \in(1, \infty) .
$$

By the corresponding result in $\mathbb{R}^{3} \times(0, T)$ of Lemma 8.1 we have $\mathbf{v} \in L^{\frac{10}{3}}\left(\mathbb{R}^{3} \times\right.$ $(0, T))$, and therefore we conclude that $p \in L^{\frac{5}{3}}\left(\mathbb{R}^{3} \times(0, T)\right)$.

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[^0]:    ${ }^{1}$ The derivative of the determinant of $N \times N$ matrix is the sum of $N$ determinants obtained from the original matrix upon substitution of each row(column) by the row (column) of the corresponding derivatives.

[^1]:    ${ }^{2} \mathrm{~A}$ more general Stress-Deformation relation is due to Serrin [39].

[^2]:    ${ }^{3}$ Recall that $\mathbf{f}$ is a specific force, that is, force per unit mass.

[^3]:    ${ }^{1}$ Osborne Reynolds, 1842-1912, Irish-born physicist, gave important contributions to the understanding of fluid Dynamics.

[^4]:    ${ }^{2}[7]$, Chap. IV, § 4

[^5]:    ${ }^{3}[7]$, Chap. IV, § 4

[^6]:    ${ }^{4}$ see 12.1. of the Complements.

[^7]:    ${ }^{5}$ See [6], Chap. 10, Theorem 1.1.

[^8]:    ${ }^{6}$ See [6], Chapter 10, Corollary 1.1.

